

MUSING ABOUT MATHEMATICS (AND TRUTH)

BARRY MAZUR

CONTENTS

1. Talking to each other	1
2. Questions in Mathematics	2
3. Before proof	2
4. Definition, Computation, Construction, Characterization, Analogy, Imagination	3
5. Why is there so much mathematics?	3
6. Essential roles that definitions play for us:	5
7. Defining Infinity:	6
8. Definition or Characterization?	8
8.1. What is a Prime Number?	8
8.2. What is a set?	9
9. If there's time... discuss	9

ABSTRACT. I want to thank Avner Ash and Ryan Hanley for inviting me to B.U. to take part in a Q&A session of their course *Morals and Mathematics*. I especially want to thank the students for bringing such a broad range of interests and experience into our conversation. Every sentence in these musings below is meant to be less *assertion* and more *question* to launch discussion.

1. TALKING TO EACH OTHER

If you think of why we humans communicate—why we talk to each other—well,

- we have a ton of practical reasons to do so; from the need to convey vital messages (e.g., *warnings about dangers*) to serving lesser necessities (*Please pass the butter*)...
- we have inner feelings, sentiments we simply want to express (*love, expression of friendship, anger, happiness, sadness*)...

- there are ‘truths’ we want to share.

Mathematical communication may be fashioned to serve any of these reasons, but mathematics seems to have a particular, and curiously intimate, way of connecting with the third—i.e., the sharing of ‘truth.’

I think this, these days, with special gratitude for the existence mathematical activity, given the daily demonstration of the fragility of our ways of transmitting truth to each other.

2. QUESTIONS IN MATHEMATICS

Once answered—or even just partially answered, as this is mostly all that happens—mathematical questions tend to expand to yield more questions, and more structure, and broader points of view.

- What is the case?
- How are we convinced that it is the case?
- What do we see more now, given that it is the case?
- What is the next question that is appropriate to ask?
- The tricky: why is it true?

These genres of questioning are listed in order of how personal they are, with the last category (i.e., *why?*) often answered by a combination of a *what* and a *how* answer that manages to provoke some (utterly personal) understanding.

3. BEFORE PROOF

Many discussions about Truth in Mathematics aim at understanding the mechanics of how mathematics deals with (and communicates by) *proof*.¹

But even prior to putting any particular piece of mathematical substance into an axiomatic setting—so, before there is any formal (or even informal) language framing it—we (and here I mean all of us, not only mathematicians) are moved by something that might be called ‘mathematical sensibility.’ We are guided by intuitions aiming toward something that might be called, simply, truth. It is this, as much as the ‘practical’ goals mentioned above, that motivates us.

¹And, of course, there is much to talk about regarding proof. Behind any mathematical argument is a stated (or, more often, implied) format of assumed axioms and rules of operation following the basic architecture of *logic*. But all this is essentially the consensual agreed-upon language for expression of mathematics, the *syntax* rather than the *substance* we focus on, when we do mathematics.

4. DEFINITION, COMPUTATION, CONSTRUCTION, CHARACTERIZATION, ANALOGY, IMAGINATION

Consider the very first construction in Euclid's *Elements*: the bisection of a line segment using ruler and compass.

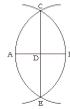


FIGURE 1. Bisecting a line interval

This construction is, of course, embedded in the armature of Euclid's 'definitions' and 'common notions' but its 'evidence to us' stems from something more primitive, I think.

To point to it by a principle—it would be the principle of *insufficient reason*: No one told you whether to swipe the compass first on the left and then on the right, or the other way around. Either way you do it you get the same answer—so this fundamental intuitive symmetry tells us that the the right half of the bisected interval is exactly the same length as the left half.

5. WHY IS THERE SO MUCH MATHEMATICS?

... and so many intriguing patterns and structures that go under the heading of *mathematics*. ... and so many numbers. Happily the first few of them, 1, 2, 3, seem to be immediate to our understanding—well, at least seem that way—and are so ubiquitously useful to us².

²Beyond these numerical companions, lies the bittersweet landscape of the Kantian *mathematical sublime* with its infinities and profundities. Happily, the "sweet" comes after the "bitter," in that, first, we bitterly face this infinite prospect: we try to grasp that ungraspable infinite with our finite minds. Only by so trying are we prepared for the sweet afterthought, that we, with our merely finite minds can miraculously manage to comprehend the impossibility of this infinite enterprise. Each of us emerges from this experience with our personal consolation prize: a "starry sky within," as Kant calls it.

And yet, why is there still so much to understand about $1, 2, 3 \dots$?

$$\boxed{2 + 1 = 3}$$

I'm pretty sure that Immanuel Kant would put the mathematical statement

$$(1) \qquad 2 + 1 = 3$$

in the category of assertions that he calls the *analytic a priori*. I know that he views the statement

$$(2) \qquad 5 + 7 = 12$$

as quite different!—he refers to the latter equation in his *Critique of Pure Reason* as an example of his notion: the *synthetic a priori*³.

The former counts as *analytic a priori* in the sense that the left-hand side of Equation (1) is simply the definition of the right-hand side.

But to understand Equation (2) you need to organize a structure of attack; e.g., in some form or other you will have incorporated (as part of your apparatus of comprehension) a version of an associative law. You need to go beyond the mere statement: you shape a mental strategy, satisfactorily go through the exercise of thought it leads you to do and... understand something.

So Equations (1) and (2) relate to different shades, if not grades, of truth.

An elementary example of *synthetic a priori*—that might capture the imagination more than Equation (2) above—is given by Proposition 32 of Book 1 of Euclid's *Elements*: the sum of the angles of a triangle is 180° :

³The word 'priori' signals that these categories are genres of thought exercised without (i.e., *prior to*) considering things other than the circle of the faculties of our mind—e.g. prior to input coming from perceptions, or more generally, from "the world"—all that being *a posteriori*.

The 'analytic a priori' is a statement that follows immediately from the very meaning of the terms; in effect, a tautology.

The 'synthetic a priori', for Kant, requires some engagement with our *intuitions*—as he calls them—space and time, these being not "out there," as one might naturally think but rather, as Kant would have it, resources of our faculties of mind: resources in which we dress objects of thought so as to be able to properly think about them.

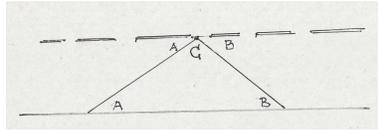


FIGURE 2

The difference with the previous example is that a *construction*—a ‘synthetic act’— launches the standard proof of this theorem. You first choose a side of the triangle as base; then draw the line parallel to that base, through the opposite vertex; and then note that the three angles (A, C and B as in the above figure) at the vertex opposite the chosen base are, in fact, equal to the three angles of the triangle. (Here you use a previous result that guarantees that for a line L that cuts a pair of parallel lines, the opposite angles created at the points of intersection of L with each of the parallel lines are equal).

There is another major difference between this proposition and Equation (2). Namely, appropriately interpreted⁴, this proposition actually *characterizes* the Euclidean plane (among all smooth not-necessarily-Euclidean ‘planes’). In a sense, the proposition can be taken to offer a *definition* of the Euclidean plane. (The Pythagorean Theorem—appropriately understood— also characterizes the Euclidean plane.)

6. ESSENTIAL ROLES THAT DEFINITIONS PLAY FOR US:

- To delineate, and help us focus on properties of interest. E.g.,

*(a curve is **convex** if...)*

- To ‘form’ mathematical objects that are to be studied. E.g.,

*(a **circle** is...)*

- To encapsulate a relationship. E.g.,

*(two triangles are **similar** if...)*

- To abbreviate.

...

⁴e.g., ‘straight lines’ are taken to be geodesics

Definitions, then, can range from lowly tautologies (e.g., Equation (1)) to illuminating puzzles—such as the definition of *infinity* or the definition of the concept of *set*.

7. DEFINING INFINITY:

Here is an exercise (for people with math backgrounds). Below I will give four possible definitions of *infinite set*—they are all minor variants of one another. (The first is due to Dedekind and resonates with that curious corridor in *Hilbert’s Hotel*.)

What do we think is the difference between these four possible definitions of *infinite set* given below?

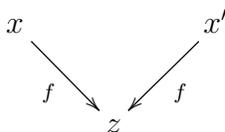
But first I’ll review two concepts: **an injective mapping** and **a surjective mapping**.

A mapping

$$f : X \longrightarrow Y$$

is called **injective** (synonym: “**one-one into**”) if it is a mapping from a set X to a set Y such that no two different elements of the set X maps to the same element of the set Y .

I.e., if for any two different elements $x \neq x' \in X$ their images under the mapping f are also different; i.e., $f(x) \neq f(x') \in Y$. That is, there is no collapsing:



Here’s an example of an injective mapping:

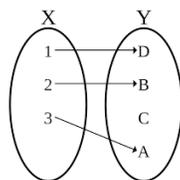


FIGURE 3. injective

A mapping

$$f : X \longrightarrow Y$$

is called **surjective** (synonym: “**onto**”) if every element $y \in Y$ is in the image of X under the mapping f ; i.e., for every element $y \in Y$ there exists an element $x \in X$ such that $f(x) = y$.

Here’s an example of a surjective mapping:

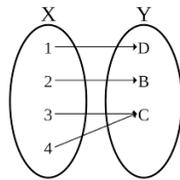


FIGURE 4. surjective

Now for the four different candidate-definitions of this concept “infinite set”:

Definition ∞_1 : A set S is **infinite** if there exists an *injective* mapping $f : S \rightarrow S$ (i.e., from the set S to *itself*) that is not surjective (equivalently: is not a one:one correspondence between the set S and itself).

Definition ∞_2 : A set S is **infinite** if there exists a *surjective* mapping $f : S \rightarrow S$ (i.e., from the set S to *itself*) that is not injective (equivalently: is not a one:one correspondence between the set S and itself).

Definition ∞_3 : A set S is **infinite** if there exists an injective mapping of the set \mathbf{N} of natural numbers into S .

(The set of natural numbers is what you think it is: $\mathbf{N} := \{1, 2, 3, 4, \dots\}$, even though the ancients were dubious about the number 1 as being in the same category as the other whole numbers. To actually *define*⁵ this set \mathbf{N} without making use of the *dot-dot-dots* requires some apparatus—e.g., mathematical induction.)

⁵And here, the late middle English sense of the word ‘define’ (to *bring to an end*) fits neatly.

Definition ∞_4 : A set S is **infinite** if there exists an surjective mapping of the set S onto \mathbf{N} .

8. DEFINITION OR CHARACTERIZATION?

Within the appropriate axiomatic set-theoretic context, the four definitions of “infinite set” are equivalent, so we have a choice:

- We can choose one of them as our primary definition, and the other three can be thought of as ‘characterizations’ of the then-defined concept—infinite set.
- We can simply say: these are all equivalent and any one can serve as “the” definition.

The relationship between these choices depend on the ambient axiomatic context in which are working. For example, if you accept the ‘Axiom of Choice’ then if a set is infinite following Definition ∞_2 it is also infinite following Definition ∞_1 .

The question, then, (*What is an infinite set?*) depends on the choice: definition versus characterization. The same holds for:

8.1. What is a Prime Number?

As for the power of definition to provide ‘focus,’ consider the two equivalent definitions of prime number (given by (1) and (2) below)—where one is left to make the choice of regarding one of these as ‘definition’ and the other as ‘characterization’:

A prime number p is a (whole) number greater than one

- (1) that is not expressible as the product of two smaller numbers.

or

- (2) having the property that if it divides a product of two numbers, it divides one of them.

If you choose (2) as the fundamental definition you are placing the notion of prime number in the broader context of ‘prime’-ness as it applies to number systems more general than the ring of ordinary numbers—and more specifically in the context of *prime ideals* of a general ring. So choosing (2) as definition casts (1) as a specific feature that characterizes prime numbers, thanks to the theorem that guarantees the equivalence of these two formulations. Going the other route—i.e., focusing on (1), the unfactorable quality of prime number,

would then cast (2) as a basic more general feature also *characterizing* prime-ness.

As for dependence on context, we might turn to the question—necessarily prior to the question *What is an infinite set?*—namely:

8.2. What is a set?

I'm not sure we have a definitive answer to this yet. A 'set' is a pretty lean mathematical object, evoked—if not captured—by the simple phrase *a collection of things*. Nevertheless *sets* provide the substrate for such a wide variety of mathematical objects. So, an axiom system that 'models' set theory is clearly of importance.

9. IF THERE'S TIME... DISCUSS

- *analogy—such as: analytic geometry— and*
- *'leaps of the imagination' in mathematics:*
 - *dealing with $\sqrt{-1}$ as a 'number,' or, for that matter,*
 - *dealing with "3" as a number.*