ORGANIZING RATIONAL THOUGHT

BARRY MAZUR

Contents

The aim of our course "PHIL 248R: Rationality" is to *live with* this notion—*rationality*—this semester, to get a better understanding of its shades of meaning in *our various areas of interest and experience and thought.*[1]

1. About the Course Requirements

As mentioned in the Syllabus:

> Students are expected to attend all classes, to keep up with the (not very onerous) reading, and to participate actively in class discussion. A term paper of about 25 pages is due by December 17. The topic must be discussed with one of the instructors and cleared by November 17. There will be no exam.

We would like students to hand in—by September 26—a very short (paragraph or two) description of the direction they are thinking of pursuing for their term paper.

2. What is rational thought?

This will be the underlying question throughout our seminar. What does it mean when we label an argument *rational*? What are the consequences of such a label? The structure of *Rational Argument* may

[1]As described in the Syllabus, we will be dealing with specific aspects of this immense subject, with emphasis strongly weighted by the interests, background, and preferences of the participants in our seminar. This is the sixth seminar-course I've taught with Amartya Sen and Eric Maskin.

Here is a link to my introductory write-up for the course we taught that was focused on '*Axiomatic Reasoning* `http://people.math.harvard.edu/~mazur/papers/Axiomatic-Reasoning.pdf`.

take a different shape in different disciplines. What can we say about this? How has the concept evolved?

Any utterance of ours—no matter how wild, and no matter in what circumstance, and even no matter what internal reflection is meant to be communicated by it—will surely have some grain of rational construction within it:

> Blow, winds, and crack your cheeks! rage! blow!
> You cataracts and hurricanoes, spout
> Till you have drench'd our steeples, drown'd the cocks!

Our focus in this session will be to review early (and also contemporary) ways of organizing modes of reasoning that might be deemed rational.

Points for Possible Discussion: Please offer any comments you have—reactions, reformulations, puzzlements, questions—regarding the issues raised in the readings assigned for today's discussion.

Is the act of *thinking rationally* appropriately described or encompassed by the formats cited in those readings? How do these formats relate to our current practices? These 'practices' may be different for different disciplines. How do they relate to the disciplines you are familiar with?

Part 1. Organizing Rational Thought

(1) A Linnaean-type categorization

$$\text{Variety} \subset \text{Species} \subset \text{Genus}$$

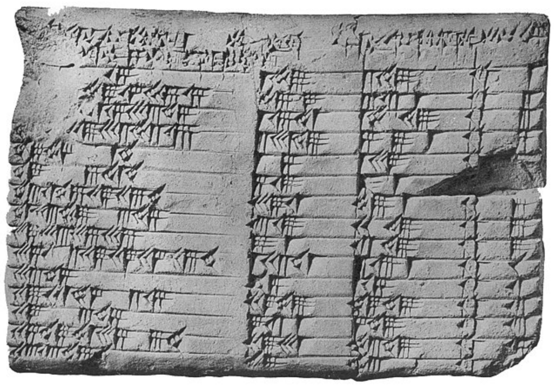
is one way of organizing our thinking about a class of objects. But even a simple **ordered list** organizes rational thought.

Legal codes, such as the ancient Babylonian Code of Hammurabi ($\sim$ 1750 BC) with its 282 rules comprises such a reasoned list.

As does the Babylonian cuneiform tablet[2] (which is even earlier : $\sim$ 1800 BC) listing some 'Pythagorean triples:' it lists (some) pairs of whole numbers (i.e., integers) such that if you build a rectangle with those numbers as the dimensions of its

[2] labelled Plimpton 322

length and width, then the length of the diagonal in such a rectangle is again a whole number.[3]



If you and I were to create such a list, we would—most likely—begin with

$$3^2 + 4^2 = 5^2$$

but the first line entry of the Plimpton 322 is:

$$119^2 + 120^2 \;=\; 169^2,$$

so the rationale for the order in this list, and the reason for creating it, and how the Babylonians got these numbers, is interesting, as is the curious fact that the Babylonians had already found arithmetic interest in the *integer* solutions[4] of—in effect—polynomial equations. There seems to be lots of controversy about this particular tablet; see the (2017) Scientific American article by Evelyn Lamb *Don't Fall for Babylonian Trigonometry Hype*[5].

Whatever type of mathematics the Plimpton tablet is, it isn't reasoned demonstration. Nor is there (at least, available to us) a trace of mathematical rational argument in any of the early Babylonian mathematical record.

[3]Equivalently, thanks to the Pythagorean Theorem, pairs of whole numbers a, b such that the sum of their squares, $a^2 + b^2$ is the square of a whole number.

[4]written in the sexagesimal system—so 119 is given as $1, 59$ (in cuneiform numerical notation)

[5]`https://blogs.scientificamerican.com/roots-of-unity/dont-fall-for-babylonian-trigonometry-hype/`

For preserved texts of *demonstration*, for the dependence of proof on *definitions, postulates* and *common notions* we may have to wait until *Euclid's Elements* some two millennia later. Mathematics has an entire vocabulary used to shape and organize arguments—*definition, lemma, proposition, theorem, conjecture, etc.* Physics similarly, including the interesting use of the label *ansatz.*

(2) Aristotle, in our readings, establishes a setting, and sketches a format that is a basis for **deliberation, discussion, argument,** and **communication of ideas**. It puts forward a vocabulary allowing us to talk about steps in rational argument.

What unmentioned assumptions are there in Aristotle's description of the 'formats of reasoning' —in the passages cited below—that may affect their effectiveness (in appropriately describing, or regulating—or even convening the meaning of—rational thought)?

(3) **A comment on the notion of** *axiom:*

- In Aristotle axioms are not specifically labelled as such, but do occur as discussion-launching points.
- For Plato, *hypothesis* plays that role just as Plato's *diairesis* plays a principle role as a mode of *definition.*
- For Euclid they are "common notions;"
- and there is the evolution:

  Euclid→Descartes→Hilbert(→Birkhoff) →*Contemporary views.*
- Consider the axiomatic formulation of issues in:
  - Social Choice, Utility Theory;
  - and the axiomatic vocabulary for models in the Sciences: Relativity Theory, and Evolution;
  - and Noam Chomsky's *Universal grammar.*
  - There's also the formulation of propositions and demonstrations in Spinoza's *Ethics.*

- The **'true and primary'** versus the **'generally accepted.'** Aristotle defines **reasoning** to be:

  > ... an argument in which, certain things being laid down, something other than these necessarily comes about through them.

  He then makes the distinction between **demonstration** and **dialectic:**

  > **It is a 'demonstration',** when the premisses from which the reasoning starts are true and primary, or are such that our knowledge of them has originally come through premisses which are primary and true.
  > Reasoning, on the other hand, is **'dialectical',** if it reasons from opinions that are generally accepted.

- The **"generally accepted"** often referred to as the **sensus communis** requires, of course, some implied *community* as the grounds for dialectic.
  It might be interesting to discuss the phrase "self-evident," as it is used, in comparison with the phrase "generally accepted,"[6]

  Aristotle deals with the peril of *contentiousness* in dialectic—in contrast to straight *mis-readings* in attempts at demonstration[7]. And the skills of rhetoric necessary for *persuasion*.

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[6]noting: Abraham Lincoln's replacement of the phrase "self-evident" by "proposition."

[7]Is an *error* in a rational argument—say, in a mathematical proof—*irrational thought*? Perhaps just call it a *bug*, saving the word *irrational* to describe a more passionate, possibly turbulent, genre of thought.

Immanuel Kant, in his essay *An Answer to the Question: "What is Enlightenment?"* (1784) writes:

> Dogmas and formulas, those mechanical instruments for rational use (or rather misuse) of his natural endowments, are the ball and chain...

of a person trying to work their way out of the immaturity of thought that has become "second nature."

- Aristotle's basic vocabulary: **proposition, problem, definition, property, genus, accident.** We'll consider each of these terms.

- **Proposition $\leftrightarrow$ Problem.** Aristotle makes a distinction here that brings to mind—-in contrast—the blurring of such a distinction in Euclid (who is writing centuries later). Consider the first four assertions in Book 1 of Euclid's Elements. These are given no labels beyond their order in the list:

  (a) On a given finite straight line to construct an equilateral triangle.
  (b) To place at a given point [as an extremity] a straight line equal to a given straight line.
  (c) Given two unequal straight lines, to cut off from the greater a straight line equal to the less.
  (d) If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

  The first three might count as 'problems" or 'constructions' while the fourth as an assertion of a proposition (following the contemporary notion of 'proposition')
  Aristotle's distinction also has a resonance with what is standard current practice in mathematics—i..e, the distinction between the (sort of formal) labels Conjecture $\leftrightarrow$ Question: a *conjecture* being a straight assertion, while a *question* having the substance of an assertion, is open-minded about whether the assertion is true or not.

- Aristotle's **classification of propositions (and problems)**:

  What do we make of this:

> Of propositions and problems there are[8] three divisions: for some are *ethical* propositions, some are on *natural philosophy*, while some are *logical*.
>
> – Propositions such as the following are *ethical*, e.g. 'Ought one rather to obey one's parents or the laws, if they disagree?';
> – such as this are *logical*, e.g. 'Is the knowledge of opposites the same or not?';
> – while such as this are on *natural philosophy*, e.g. 'Is the universe eternal or not?'

- **Definition... and definitory... as related to:**
  property, essence, genus, accident

> A 'definition' is a phrase signifying a thing's essence. It is rendered in the form either of a phrase in lieu of a term, or of a phrase in lieu of another phrase; for it is sometimes possible to define the meaning of a phrase as well... One may, however, use the word 'definitory' also of such a remark as 'The "attractive" is "beautiful", and likewise also of the question, 'Are sensation and knowledge the same or different?'

It would be good to compare this concept of *definition* with various definitions in Euclid's Elements; and also with the somewhat different meaning (and format) of the word *definition* in modern mathematics[9].

First, Euclid's first three definitions of Book 1 of definitions:

[8] "to comprehend the matter in outline." writes Aristotle

[9] and even earlier; for example Spinoza— in his essay *On the Improvement of the Understanding*—gives, among his four rules for defining (as he puts it: "uncreated"—i.e., mathematical) objects:
– When the definition of the thing has been given, there must be no room for doubt as to whether the thing exists or not.
– Though this is not absolutely necessary, it should be possible to deduce from the definition all the properties of the thing defined.

– A point is that which has no part.
– A line is breadthless length.
– The extremities of a line are points.

Compare this with Definition 26 of Book 11 of Euclid's Elements. This has quite a different flavor (*we can discuss this*).

– An octahedron is a solid figure contained by eight equal and equilateral triangles.

Perhaps the first three definitions are—to use Aristotle's term–definitory: the give a sense of the essence of the notion—while the definition of 'octahedron' could actually be used in a construction and demonstration.

The mathematician David Hilbert rewrote Euclid's foundations. He introduces *point* and *line* as *undefined terms*—calling them 'things'—where their meaning results only from the axioms describing their relationship to each other. Hilbert's rewriting of Euclid's *Elements* begins:

> Let us consider three distinct systems of things. The things composing the first system, we will call **points** and designate them by the letters A, B, C,. . . ; those of the second, we will call **straight lines** and designate them by the letters a, b, c,. . . The points are called the **elements of linear geometry**; the points and straight lines, **the elements of plane geometry**. . .

Hilbert allows himself these undefined terms: *point, line, plane, lie, between,* and *congruence* Euclid's definitions of *point* and *line* seem to be whittling these concepts into their pure form from some more materially graspable context (e.g., where lines have breadth)[10] while for Hilbert the essence of *point, line, plane, lie, between,* and *congruence* is simply their relationship one to another.
Once one allows the bedrock of—say—Set Theory, definitions are often 'delineations of structure,' cut out by means of quantifiers and predicates but making use of set theoretic, or at least priorly defined objects. E.g. *A circle is*

[10]I want to thank Eva Brann for pointing this out.

a set of points equi-distant from a single point in the Euclidean plane. Compare these with modern definitions of mathematical concepts—such as Richard Dedekind's definition of *infinite set.*

- **Essence, {defining property}, characterization, property, accident**
  Discuss this hierarchy... including the choices allowable to one in mathematics; e.g., a **positive real number** is—

  - a number greater than zero. or:
  - the square of a nonzero positive number.

  You choose! Take one as definition and the other becomes a *characterizing property* of the notion: **positive real number**. Take the other as definition, and ditto: the first becomes a *characterizing property* of the notion. Or, you can be evenhanded and simply say that each of them is 'a defining property' of the notion. These choices may be logically equivalent but the second definition is given by an "algebraic" condition—making the notion dependent on the algebra of the surround, while the second definition is dependent on the order relation of the surround[11].

- **Logic** as in the *Prior Analytics* and *Posterior Analytics* of Aristotle frame the scaffolding on which we build arguments and justify them. For example, in Book I of the *Prior Analytics* Aristotle defines what he refers to as a *syllogism*:

  > A syllogism is an argument (*logos*[12]) in which, certain things being posited, something other

[11] Mathematical choices that are equivalent from a purely logical perspective, but nevertheless change one's viewpoint are abundant. Consider this comment of Emmy Noether (one of the great early twentieth century mathematicians—a founder of modern algebra):

> If one proves the equality of two numbers a and b by showing first that $a \geq b$ and then $b \geq a$ it is unfair; one should instead show that they are really equal by disclosing the inner ground for their equality.

[12] But see Stephen Read's commentary on the translation of the word *logos* as 'argument' in this quotation: `https://www.st-andrews.ac.uk/~slr/The_Syllogism.pdf`

> than what was laid down results by necessity
> because these things are so. (24b19-20)

Mathematical logic in its more contemporary dress is a direct development of this. But we can discuss how it differs! David Hilbert, again, (∼1921) framed the notion of **formal system**, its architecture being appropriate to serve as the language of rigorous mathematical arguments. Even though a formal system is meant to be—in effect—a language within which one can frame arguments, it itself is formulated as a mathematical concept in its own right.

A **formal system** consists of

- *A finite set of symbols*, (the alphabet)
- *Formulas* which are finite strings of symbols taken from the alphabet,
- *A grammar* consisting of rules to form formulas from simpler formulas. A formula is said to be **well-formed** if it can be formed using the rules of the formal grammar.
- *A set of axioms* consisting of well-formed formulas, and:
- *A set of inference rules.*

The basic 'rule of inference' beyond rules that govern the use of various logical operations is *modus ponens*; in effect: the syllogism.

- **Descartes; Rationality and 'method.' The training of mind, the marshaling of intuitions, 'certainty'**

We will surely not have enough time to discuss this as fully as it deserves, but... what do we think of his first few rules?

> (a) The aim of our studies should be to direct the mind with a view to forming true and sound judgements about whatever comes before it.
>
> (b) We should attend only to those objects of which our minds seem capable of having certain and indubitable cognition.
>
> But one conclusion now emerges out of these considerations, viz. not, indeed, that

Arithmetic and Geometry are the sole sciences to be studied, but only that in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry.

(c) Concerning objects proposed for study, we ought to investigate what we can clearly and evidently intuit or deduce with certainty, and not what other people have thought or what we ourselves conjecture. . . .

(d) We need a method if we are to investigate the truth of things.

(e) The whole method consists entirely in the ordering and arranging of the objects on which we must concentrate our mind's eye if we are to discover some truth. We shall be following this method exactly if we first reduce complicated and obscure propositions step by step to simpler ones, and then, starting with the intuition of the simplest ones of all, try to ascend through the same steps to knowledge of all the rest.

Part 2. **Relatively Early Attitudes toward Experiment**

3. What is an experiment?

We might consider the label "experiment" as fitting comfortably in a much broader range of activities; for example:

(1) *Pure observation of the natural world.*

This category includes the work of naturalists; also careful visual renditions, as in James Audubon's *Birds of America*; e.g:

.

Also records in the style, say, of Aristotle's description of the generation of chicks from eggs[13]:

> Generation from the egg occurs in an identical manner in all birds, though the time taken to termination varies, as we have said. In the case of the hen, the first signs of the embryo are seen after three days and nights; in larger birds it takes more time, in smaller birds less. During this time the yolk travels upwards to the point of the egg—that is where the starting point of the egg is and where it opens up, and the heart is no bigger than just a small blood-spot in the white. This spot beats and moves as though it were alive; and from it, as it grows, two vein-like vessels with blood in them lead on a twisted course to each of the two surrounding membranes. A membrane with bloody fibers already surrounds the white of the egg, at this time coming from the vessel-like channels. A bit later the body can also be distinguished, at first very small and pale. The head is apparent, and its eyes, very swollen; and this continues for a long time, for it is later that they contract and become smaller.

(2) *Observation meant to resolve a question.*

In the passages of Francis Bacon's Novum Organum that was in the readings, Bacon presses for a mode of (scientific) experiment that:

> though difficult in its operation, is easily explained. It consists in determining the degrees of certainty, while we, as it were, restore the senses to their former rank, but generally reject that operation of the mind which follows close upon the senses, and **open and establish a new and certain course for the mind from the first actual perceptions of the senses themselves**.

Bacon claims: "But the manner of making experiments which men now use is blind and stupid."

[13] De Generatione Animalium (561a4-21)

He categorizes the various ways in which one is moved to (self-) deception in rational thought, (his term: "idols").

And then Bacon describes a series of experiments in Book 2 of Novum Organum. He poses a series of questions, with suggestions for 'experiments' to answer those questions: "Let there be an experiment" is the repeated phrase.

The answers he seeks has to do with something he calls the **forms** that demonstrate the unity of particular 'things.' His use of the word *forms* resonates with Plato's $\epsilon\iota\delta\eta$ and/or the Aristotelian notion usually translated as *formal cause*.

Here's Bacon:

> But he who knows forms grasps the unity of nature beneath the surface of materials which are very unlike. Thus is he able to identify and bring about things that have never been done before, things of the kind which neither the vicissitudes of nature, nor hard experimenting, nor pure accident could ever have actualised, or human thought dreamed of. **And thus from the discovery of the forms flows true speculation and unrestricted operation**.

A typical example (taken from a list of his suggestions to 'investigate the *forms of heat*'):

> Spices and acrid herbs are sensibly warm to the palate, and still more so when taken internally; one should see, therefore, on what other substances they exhibit the effects of heat. Now, sailors tell us that when large quantities of spices are suddenly opened, after having been shut up for some time, there is some danger of fever and inflammation to those who stir them or take them out.**An experiment might, therefore, be made** whether such spices and herbs, when produced, will, like smoke, dry fish and meat hung up over them.

As I read Bacon's Novum Organum, it seemed curious to me how little one sees the rigid format that one often hears: the **straight Baconian model** view of an empirical investigation—

Set-up and Hypotheses $\longrightarrow$ Data Collecting $\longrightarrow$ Processing Data and Conclusion.

where one begins with an explicitly described 'hypothesis' and the aim is to perform an experiment to decide whether the hypothesis is confirmed or shown to be faulty. If the former, one has evidence for the truth of the hypothesis; if the latter, one rejects the hypothesis, following the framework of Richard Feynman's historical comment:

> It doesn't make any difference how beautiful your guess is, it doesn't make any difference how smart you are, who made the guess, or what his name is. If it disagrees with experiment, it's wrong. That's all there is to it[14].

(3) *Galileo: Experiments tested materially, and Thought-Experiment*

A generation after Bacon, Galileo in his *Dialogue Concerning Two New Sciences*[15] (1638) describes experiments (one of them resonating with the much later thought-experiments of Einstein).

- The first experiment begins with the famous comment on Aristotle's way of handling the issue of velocity of falling stones (this is uttered by Salvatio in the Dialogue):

  > SALV: I greatly doubt that Aristotle ever tested by experiment whether it be true that two stones, one weighing ten times as much as the other, if allowed to fall, at the same instant, from a height of, say, 100 cubits, would so differ in speed that when the heavier had reached the ground,

[14] But, a Physicist friend (Melissa Frankin) told me that in her experience, experiments often are not exactly conducted with confirmation or rejection of some explicit hypothesis in mind even though their written accounts tend to be phrased in that format.

[15] (What are those *two* Sciences?)

> the other would not have fallen more than 10 cubits.

This is followed by Simplico's assertion of trust in Aristotle's claims: if Aristotle says "*we see*" we trust him:

> SIMP: His language would seem to indicate that he had tried the experiment, because he says: We see the heavier; now the word see shows that he had made the experiment.

This, then, is answered by Sagredo—who has tested the claim:

> SAGR: But I, Simplicio, who have made the test can assure you that a cannon ball weighing one or two hundred pounds, or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits.

and this is followed by Salvatio describing *in detail* the (celebrated) thought-experiment that would lead to the same conclusion as Sagredo's test.

- The second experiment is the excerpt in our reading: the question of speed of light (this being a pure thought-experiment):

> SIMP:
> Everyday experience shows that **the propagation of light is instantaneous**; for when we see a piece of artillery fired, at great distance, the flash reaches our eyes without lapse of time; but the sound reaches the ear only after a noticeable interval.
>
> SAGR:
> Well, Simplicio, the only thing I am able to infer from this familiar bit of experience is that sound, in reaching our ear, travels more slowly

> than light; it does not inform me whether the coming of the light is instantaneous or whether, although extremely rapid, it still occupies time. An observation of this kind tells us nothing more than one in which it is claimed that "As soon as the sun reaches the horizon its light reaches our eyes"; but who will assure me that these rays had not reached this limit earlier than they reached our vision?
>
> SALV:
> The small conclusiveness of these and other similar observations once led me to devise **a method by which one might accurately ascertain whether illumination, i. e., the propagation of light, is really instantaneous.** The fact that the speed of sound is as high as it is, assures us that the motion of light cannot fail to be extraordinarily swift. The experiment which I devised was as follows: Let each of two persons take a light contained in a lantern, or other receptacle, such that by the interposition of the hand, the one can shut off or admit the light to the vision of the other. Next let them stand opposite each other at a distance of a few cubits and practice until they acquire such skill in uncovering and occulting their lights that the instant one sees the light of his companion he will uncover his own.

Question 3.1. What is a "Thought-Experiment?"

(4) **Causality** The notion of *causality* as a topic for discussion. E.g.: From David Hume's *Treatise on Human Nature*:

> To begin with the first question concerning the necessity of a cause: It is a general maxim in philosophy, that whatever begins to exist, must have a cause of existence. This is commonly taken for granted in all reasonings, without any proof given or demanded. It is supposed to be founded on intuition, and to be one

> of those maxims, which though they may be denyed with the lips, it is impossible for men in their hearts really to doubt of.

Many experiments are meant to offer conclusions of the form:

A is the cause of B

where the meaning of the word "cause" is left to be self-evident. Is the word "Because" simply a tag—a sort of noble lie—that we all use to claim justification of an assertion?

Or is it *denknotwendig* (necessary for thought) as Kant felt?

Undefined, or even undefinable, "causal relations" are unavoidable. Anything we write, if it is to have some coherent arc[16] has to have some semblance of causal structure. In any narrative of the form:

> The Queen died. Then the King died.

we expect that the "Then" hints at a cause (other than chronological order)...e.g., perhaps:

> The Queen died. Then the King died *of grief*.

Question 3.2. What is the format—or vocabulary—of causal conclusions in the specific subjects that you study, subjects that you are at home with?

Question 3.3. In what sense does a proof in mathematics that assumes A in order to prove B constitute a causal relation between the A and the B?

Take a look at the first few pages of an essay I wrote discussing this:

On the word "because" in mathematics, and elsewhere

(`https://canvas.harvard.edu/courses/108721/files?preview=15865181`).

[16] for it not to be a recital of "one damn thing after another"

Also, what do you make of this baffling comment of Aristotle (in Part 9, Book II of The Physics)?

> Necessity in mathematics is in a way similar to necessity in things which come to be through the operation of nature. **Since a straight line is what it is, it is necessary that the angles of a triangle should equal two right angles.** But not conversely; though if the angles are not equal to two right angles, then the straight line is not what it is either.

(5) *Correlation* and *Causation*. How do you understand the *correlation* versus *causation* issue? For example, is it the case that, thanks to the IT revolution, it is so easy for us to amass masses of data, that we have little or no need for explanatory models dealing with causality?

This is the thrust of the debate between Noam Chomsky and Peter Norvig `http://norvig.com/chomsky.html`

4. Forms of experiment, and ways of accumulating data

(1) **Randomized Controlled Trials (RCT)**

An Example

Objective: To assess the effectiveness of pilates method on patients with chronic non-specific low back pain (LBP).

Method: A randomized controlled trial was carried out in sixty patients with a diagnosis of chronic non-specific LBP. Patients were randomly assigned to one of two groups: Experimental Group (EG) that maintained medication treatment with use of NSAID and underwent treatment with the pilates method and Control Group (CG) that continue medication treatment with use of NSAID and did not undergo any other intervention. A blinded assessor performed all evaluations ...

Results: The groups were homogeneous at baseline. Statistical differences favoring the EG were found with regard to pain ($P < 0.001$), function ($P < 0.001$) and the quality of life domains of functional capacity ($P < 0.046$)...

Statistical differences were also found between groups regarding the use of pain medication with the EG taking fewer NSAIDs than the CG.

Conclusions: The pilates method can be used by patients with LBP to improve pain, function and aspects related to quality of life (functional capacity, pain and vitality). Moreover, this method has no harmful effects on such patients.

RCT is often called the "gold standard" for the format of experiments—and this is debated in various circles at present. See *Why the 'gold standard' of medical research is no longer enough* published in StatNews by Tom Frieden, a former director of CDC:

> The emerging use of "big data," including information from electronic health records and expanded patient registries, presents new opportunities to conduct large-scale studies with many of the benefits of RCTs but without the expense.

(2) **Natural Experiment**

Definition 4.1. A natural experiment is an empirical study in which individuals (or clusters of individuals) are exposed to the experimental and control conditions that are determined by nature or by other factors outside the control of the investigators.

An Example: smoking ban

In Helena, Montana a smoking ban was in effect in all public spaces, including bars and restaurants, during the six-month period from June 2002 to December 2002. Helena is geographically isolated and served by only one hospital. The investigators observed that the rate of heart attacks dropped by 40% while the smoking ban was in effect.

Opponents of the law prevailed in getting the enforcement of the law suspended after six months, after which the rate of heart attacks went back up.

This study was an example of a natural experiment, called a **case-crossover experiment,** where the exposure is removed for a time and then returned. The study also noted its own weaknesses which potentially suggest that the inability to control variables in natural experiments can impede investigators from drawing firm conclusions.

(3) **'Educating one's Priors.' (The bayesian format)**

The manner in which one proceeds from data to conclusion is often understood to be a straight comparison of what the hypotheses would predict and what the data reveals[17], the comparison being (usually) quantitative with a pre-specified tolerance of discrepancy (between prediction and observation).

All this is significantly modified by the Bayesian viewpoint, which methodically intertwines the first two steps, and has a different take on each of these ingredients: hypothesis, data, conclusions. We'll discuss this below[18]. We'll look at the Bayesian viewpoint as offering a 'model' to help us understand, and deal with, the interplay between those ingredients. Let's call it the **Bayesian model** for a scientific investigation.

A further issue that complicates the contrast of *models of getting to scientific conclusions* alluded to above is the difference between the Bayesian's and the Frequentist's work; their methods are not the same, and they have slightly different *primary goals.*

5. Prior information and the Birthday problem

To introduce ourselves to this 'Bayesian intertwining' (taking as a **black box**—at least at first—some of the mathematical procedures involved) let's revisit a famous problem: the birthday problem. You have a class of fifth graders in an elementary school. Suppose there are 23 students in the class. What is the probability that two of them have the same birthday? Or, to

[17] although it might be difficult to find this expressed in Bacon's writings as bluntly

[18] A disclaimer: I know very little statistics; I'm a total outsider to this field and especially to the extended conversation—and the somewhat sharp disagreements—that Bayesians and Frequentists have.

seem more mathematical, suppose there are n students. What is the answer as a function of n?

Here is the simple naive analysis of this problem. We assume, of course, that the probability of anyone having a birthday at any specific day, e.g., April 22, is 1/365 (ignoring the leap year issue). Think of the teacher marking off—successively— on a calendar the birthdays of each student. We are going to gauge the possibility that in his class of n students there are no two birthdays on the same calendar day. The first student's birthday is duly marked. We can't possibly have a concurrence of birthdays (call it a *hit*) at this point, there being only one mark. So we can record "1" as the probability that we didn't get a *hit* at least so far[19].

As for the second student, the probability of him or her not having a birthday on the same day as student #1—i.e., that there not be a *hit*— is

$$1 - \frac{1}{365} \quad = \quad \frac{364}{365}.$$

Given this situation, and passing to the third student, in order for there not to be a hit, his or her birthday has to avoid two days, so that probability is

$$1 - \frac{2}{365} \quad = \quad \frac{363}{365}.$$

Putting the two probabilities together we get that–so far in our count—the probability that there isn't a hit with these three students is

$$(1 - \frac{1}{365})(1 - \frac{2}{365}) \quad = \quad (\frac{364}{365}) \cdot (\frac{363}{365}).$$

Working up (by mathematical induction) the probability that there's no hit, with n students is then:

[19] We are going to write probabilities as numbers between 0 and 1. So if the probability of an event is $\frac{1}{2}$ that's the same as saying that it is *even odds of it happening or not happening* or that 50% *of the time it happens*, or one sometimes simply says that there's is a 50/50 chance of it occurring.

$$(1-\frac{1}{365})(1-\frac{2}{365})\cdots(1-\frac{n-1}{365}),$$

which when $n = 23$ is close to $\frac{1}{2}$. That is, for a class of 23 students the chances are 50/50 that there's a concurrence of birthdays—given this analysis.

My Bayesian friend Susan Holmes tells me that she has actually tried this out a number of times in real live classes, and discovered that the odds seem to be much better than 50/50 for 23 students; you even seem to get 50/50 with classes of as low as 16 students.

There is something too naive in the analysis above, says Susan. We should, at least, make the following (initial) correction to our setting-up of the problem. We said above:

> We assume, of course, that the probability of anyone having a birthday at any specific day, e.g., April 22, is 1/365

BUT we actually *know* stuff about the structure of our problem that we haven't really registered in making that assumption.

For example, it is a class of fifth-graders so, chances are, they were all (or mostly) born in the same year. In particular, the years of their birth all (or mostly) had the same weekends and weekdays. In the era of possible c-sections and induced births—given that doctors and hospital staff would prefer to work on weekdays rather than weekends—one might imagine that the probability of being born on a weekday is somewhat skewed. We also know more that might make us think that fixing 1/365 at the rate is too naive.

Perhaps then, instead of sticking to the probability $p = 1/365$ per day hypothesis, allow a bit of freedom and a priori allow that there are different probabilities

$$p_1, p_2, p_3, \cdots, p_{365}$$

for each day of the year[20], about which we can make very very rough guesses.

But let us not write this in stone yet. Make a mildly educated guess of these p_i; e.g., if "i" is a Saturday or Sunday

[20] these summing to 1

(or a holiday), then p_i is probably slightly less than 1/365; if a weekday, slightly more.

This initial guess (of the values of $p_1, p_2, p_3, \cdots, p_{365}$) we'll call a **Prior**. From any prior we can deduce—essentially by a straight computation as we did above with the "constant prior: 1/365"—all the expected odds and whatever statistics one wants.

BUT we have hardly gotten our best answer! All these p_i's constituted, after all, just our very very rough guess based on some intuitive hunch, prior to having any hard data.

Computing with these p_i's gives us a "number" as output—perhaps more accurate than the 23 we started this discussion with, but how does this number **compare with the actual numbers we're actually accumulating** by sampling birthday statistics for classes of fifth-graders?

The Bayesian will use this accumulating *Data* to "correct" the prior (guessed) probabilities p_i, to be more in tune with the accumulating data.

This is what I mean by the *Bayesian intertwining*: the data–as it comes in–is used to "educate the prior."

And this educated-prior is called (naturally) a **posterior**.

In some sense, the principal role of *data* in this Bayesian model is *to be fed back into the prior to refine it* to produce successive posteriors rather than (with a straight up or down judgment) to verify or contradict an hypothesis.

Starting anew with the latest **posterior** rather than the original **prior** we can deduce—essentially as we did above with the "constant 1/365" or any prior **prior** —all the expected odds and whatever statistics one wants.

In fact, there are no firm hypotheses within the Bayesian model, and no firm conclusions. I said, though: "*within* the Bayesian model." You might think that this merely produces a never-ending loop.

Nevertheless from this procedure one might extract a conclusion, but this is outside the format.

This set-up is a preliminary move in the Bayesian direction, but we aren't quite there yet. Another–and better–way of viewing this move (reflecting our most up-to-date version of belief about the set-up) is that the initial values

$$p_1, p_2, p_3, \cdots, p_{365}$$

should not be taken as hard unchangeable numbers but rather are to be viewed *each* as a"random variable" in its own right, and is subject to its own distribution (of values), which we are bent on determining, given enough **Data**.

The grand function of the data is to be fed back to educate the prior but retaining its status as probabilities.

The structure, then, is:

$$\text{Prior (probabilities)} \quad \xrightarrow{\textbf{Data}} \quad \text{Posterior (probabilities).}$$

The **black box**—so far—is that I have not yet said anything about the mathematical procedure Bayesians use to feed back (as an afterburner) information obtained by the Data into the prior assumptions, in order to effect the "education" of these prior assumptions and thereby produce the posterior. For the moment—in this discussion—it is more important for me simply to emphasize that *whatever this procedure is* it is, in fact, a *predetermined procedure.*

6. Predesignation versus the self-corrective nature of inductive reasoning

Now you might well worry that this Bayesian ploy is like curve-fitting various hypotheses[21] to the data—a kind of hypothesis-fishing expedition, if you want. You keep changing the entire format of the problem, based on accumulating data. The Bayesians have, as I understand it, a claim: that any two 'reasonable' priors, when "corrected" by enough data will give very close posteriors. That is, the initial rough-hewn nature of the prior will iron out with enough data. Their motto:

[21]I want to use the word *hypothesis* loosely, for the moment; that is, the way we generally use the word; and not in the specific manner that statisticians use it.

> *Enough data swamps the prior.*

I've been playing around with another formulation of that motto:

> *Any data-set is, in fact, a 'data point' giving us information about the probability distribution of priors.*

In contrast, there is a motto that captures the sentiment of a Frequentist:

> *Fix hypotheses. This determines a probability distribution to be expected in the data. Compute data. If your hypotheses are good,* **in the limit** *the data should conform to that probability distribution.*

About the above, one of the early great theorizers in this subject (and specifically regarding probability, randomness, and the law of large numbers) was Jacob Bernoulli. He *also* was a theologian preaching a specifically Swiss version of Calvinism. You see the problem here! There is a strict vein of *predetermined* destiny or fatalism in his theology, someone who is the father of the theory of randomness. How does he reconcile these two opposites? Elegantly, is the answer! He concludes[22] his treatise *Ars Conjectandi*, commenting on his law of large numbers, this way:

> Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of

[22] It is, in fact, the conclusion of the *posthumously* published treatise (1713) but it isn't clear to me whether or not he had meant to keep working on the manuscript.

> the universal return of things to their former positions [apokatastasis], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state.

Apokatastasis is a theological term, referring to a return to a state before the fall (of Adam and Eve)[23].

Also, we might connect the above with C.S. Peirce's 1883 paper "A Theory of Probable Inference"[24]. Peirce makes a distinction between *statistical deduction* and *statistical induction* the first being thought of as reasoning from an entire population to a sample, and the second being reasoning from sample to population. In the first it is a matter of long run frequency (i.e, the Frequentist's motto) whereas the second is related to a Peircean conception of *the self-corrective nature of inductive reasoning* (and this sounds like the Bayesian protocol).

Peirce dwells on the issue of *predesignation* in the Frequentist's context (i.e., you fix a model and then collect evidence for or against it; you don't start changing the model midstream in view of the incoming evidence). As already mentioned, there is a curious type of *meta-predesignation* in the Bayesian context, in that the manner in which you change the model, given incoming evidence, is indeed pre-designated.

Extending this, one might think of any (pre-designated) recursive format that provides successive approximations to a sought-for limit as something of an allegory of the Bayesian viewpoint.

[23]Noah Feldman once suggested to me that Calvinists might be perfectly at home with random processes leading to firm limiting fatalism, in that the fates of souls—in Calvinist dogma—are *randomly assigned* and not according to any of their virtues; i.e., to misquote someone else: "goodness had nothing to do with it."

[24]For a readable discussion of this paper, see: Len O'Neill's *Peirce and the Nature of Evidence* published in the Transaction of the Charles S. Peirce Society **29** Indiana Univ. Press (1993) pp. 211-224.

Part 3. The Inner Experience of Rationality

In the readings for today we see the experience of *rational understanding* described as being connected with—and even defined in terms of—a wide variety of feelings, such as surprise; a sense of certainty; a sense of ease; as engaging with intuitions; (intuitions meant differently by different authors); as passions for unhindered fluency thought. We also read that (for William James) 'rationality' is captured by the absence of the irrational.[25]

Lots of food for discussion!

7. "I understand!"

We all have had that feeling, from time to time: some idea or constellation of ideas–initially obscure is—(*surprise:* suddenly)—illuminated.

Archimedes had his *Eureka* (his "I found it").

Teachers try to instill this feeling—at least in specific instances—in their students.

(What is the nature of such experiences?—are there areas of thought—or kinds of reflection— that require such moments of understanding in order to proceed to a deeper level of comprehension? Are there others that don't require them?)

Our reading for today's session included Poincaré describing his (now famous) *Aha moment*[26]:

> ... The incidents of the travel made me forget my mathematical work[27]. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have

[25]The parts of text in wider margin in this handout are, for the most part, straight excerpts from readings assigned for today. The format (organization of paragraphs, italics and boldface), though, is mine—meant to to focus on those parts of the text that might prompt themes for our seminar discussion.

[26]Such an 'Aha' is a more dramatic version of the Greek Stoic idea that before a perception imprints itself (tuposis) for it to actually be 'fit to be grasped' you have to—internally, conscious or not— check off some sort of 'assent' (synkatathesis).

[27]I don't believe this at all!

> paved the way for it,[28] that *the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry.* I did not verify the idea; I should not have had time, as, upon taking my set in the omnibus, I went on with a conversation already commenced, **but I felt a perfect certainty.** On my return to Caen, for conscience' sake, I verified the result at my leisure.
>
> Then I turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the seaside and thought of something else. One morning, while walking on the bluff, the idea came to me, **with just the same characteristics of brevity, suddenness and immediate certainty,** that *the arithmetic transformations of indefinite ternary quadratic forms were identical with those of non-Euclidean geometry.*

8. Chinese room thought experiment

We... sometimes understand things. 'Other people' understand things. Dogs cats, dolphins, elephants understand things. Is the following an example of genuine *understanding*?

From Wikipedia:

Searle's thought experiment begins with this hypothetical premise: suppose that artificial intelligence research has succeeded in constructing a computer that behaves as if it understands Chinese. It takes Chinese characters as input and, by following the instructions of a computer program, produces other Chinese characters, which it presents as output.

Suppose, says Searle, that this computer performs its task so convincingly that it comfortably passes the **Turing test**:

it convinces a human Chinese speaker that the program is itself a live Chinese speaker.

[28] nor this!

To all of the questions that the person asks, it makes appropriate responses, such that any Chinese speaker would be convinced that they are talking to another Chinese-speaking human being.

The question Searle wants to answer is this:

does the machine literally "understand" Chinese?

Or is it merely

simulating the ability to understand Chinese?

Searle calls the first position **"strong AI"** and the latter **"weak AI"**.

Searle then supposes that he is in a closed room and has a book with an English version of the computer program, along with sufficient papers, pencils, erasers, and filing cabinets.

Searle could receive Chinese characters through a slot in the door, process them according to the program's instructions, and produce Chinese characters as output, without understanding any of the content of the Chinese writing.

If the computer had passed the Turing test this way, it follows, says Searle, that he would do so as well, simply by running the program manually.

Searle asserts that there is no essential difference between the roles of the computer and himself in the experiment. Each simply follows a program, step-by-step, producing behavior that is then interpreted by the user as demonstrating intelligent conversation.

However, Searle himself would not be able to understand the conversation. ("I don't speak a word of Chinese," he points out.)

Therefore, he argues, it follows that the computer would not be able to understand the conversation either.
Searle argues that, without "understanding" (or "intentionality"), we cannot describe what the machine is doing as "thinking" and, since it does not think, it does not have a "mind" in anything like the normal sense of the word.

Therefore, he concludes that the "strong AI" hypothesis is false.

9. Framing

Here's David Hume in Section 2 (On the Origin of Ideas) of *An Inquiry Concerning Human Understanding*—

> All ideas, especially abstract one, are naturally faint and obscure. . .
>
> On the contrary, all impressions, that is, all sensations either outward or inward, are strong and vivid. . . nor is it easy to fall into any error or mistake with regard to them.

Well. . .

As in Professor Maskin's presentation on the work of Tversky- Kahneman, **our different frames**[29] **of mind** (pun semi-intended) organize *or distort* our thoughts (e.g., Kahneman's "Thinking Fast and Slow"). And even if we are rationally aware of the possibility of distortion (in thinking) we find it difficult to correct these distorted perceptions or thought processes. . . as in these illusions:

(The yellow lines in the first figure are equal in size on the page, and so are the two intervals in the second picture.)

10. The 'psychology" of rationality

This is from William James' The Sentiment of Rationality.[30].

- **Rationality is recognized by certain subjective marks**.

[29]A striking Tversky-Kahneman-type experiment in 'framing':
separating a class of students into two groups, and putting them in different rooms, you ask the students in one room to estimate—not calculate— the number

$$1 \cdot 2 \cdot 3 \cdot \cdots \cdot 9 \cdot 10$$

(this is asked verbally, so the "1" comes first and the "10" comes last;

and ask the students in the other room to estimate—not calculate— the number

$$10 \cdot 9 \cdot 8 \cdot \cdots \cdot 2 \cdot 1.$$

The average 'estimate' offered by students from the first room turned out to be significantly lower than the average of the second room. . . and both were smaller than the actual number: $3,628,800$.

[30]`https://canvas.harvard.edu/courses/108721/files?preview=15979435`

What, then, are the marks of 'rationality'? A strong feeling of ease, peace, rest, is one of them. The transition from a state of puzzle and perplexity to rational comprehension is full of lively relief and pleasure.

- **Via Negativa**
  But this relief seems to be a negative rather than a positive character. Shall we then say that the feeling of rationality is constituted merely by the absence of any feeling of irrationality? I think there are very good grounds for upholding such a view.

- **Fluency**
  As soon, in short, as we are enabled from any cause whatever to think with perfect fluency, the thing we think of seems to us pro tanto rational. Whatever modes of conceiving the cosmos facilitate this fluency, produce *the sentiment of rationality.* Conceived in such modes, being vouches for itself and needs no further philosophic formulation. But this fluency may be obtained in various ways; and first I will take up the theoretic way.

- **Simplicity**
  The facts of the world in their sensible diversity are always before us, but our theoretic need is that they should be conceived in a way that reduces their manifoldness to simplicity. Our pleasure at finding that a chaos of facts is the expression of a single underlying fact is like the relief of the musician at resolving a confused mass of sound into melodic or harmonic order. The simplified result is handled with far less mental effort than the original data; and a philosophic conception of nature is thus in no metaphorical sense a labor-saving contrivance.

- **The passion for parsimony, for economy**
  of means in thought, is the philosophic passion par excellence; and any character or aspect of the world's phenomena which gathers up their diversity into monotony will gratify that passion, and in the philosopher's mind stand for that essence of things compared with which

all their other determinations may by him be overlooked.

- **More universality or extensiveness**
  is, then, one mark which the philosopher's conceptions must possess. Unless they apply to an enormous number of cases they will not bring him relief. The knowledge of things by their causes, which is often given as a definition of rational knowledge, is useless to him unless the causes converge to a minimum number, while still producing the maximum number of effects. The more multiple then are the instances, the more flowingly does his mind rove from fact to fact. The phenomenal transitions are no real transitions; each item is the same old friend with a slightly altered dress. Who does not feel the charm of thinking that the moon and the apple are, as far as their relation to the earth goes, identical; of knowing respiration and combustion to be one;

- **Particularity**
  But alongside of this passion for simplification there exists a sister passion, which in some mind—though they perhaps form the minority—is its rival. This is the passion for distinguishing; it is the impulse to be acquainted with the parts rather than to comprehend the whole. Loyalty to clearness and integrity of perception, dislike of blurred outlines, of vague identifications, are its characteristics. It loves to recognize particulars in their full completeness, and the more of these it can carry the happier it is. It prefers any amount of incoherence, abruptness, and fragmentariness (so long as the literal details of the separate facts are saved) to an abstract way of conceiving things that, while it simplifies them, dissolves away at the same time their concrete fulness. Clearness and simplicity thus set up rival claims, and make a real dilemma for the thinker.

- **A balance between these two cravings:**
  The fate of Spinoza, with his barren union of all things in one substance, on the one hand; that of Hume, with

his equally barren 'looseness and separateness' of everything, on the other,—neither philosopher owning any strict and systematic disciples to-day, each being to posterity a warning as well as a stimulus,—show us that the only possible philosophy must be a compromise between an abstract monotony and a concrete heterogeneity.

- **But the only way to mediate between diversity and unity**

  is to class the diverse items as cases of a common essence which you discover in them. Classification of things into extensive 'kinds' is thus the first step; and classification of their relations and conduct into extensive 'laws' is the last step, in their philosophic unification.

- **Familiarity, Custom, Habit**

  Philosophers long ago observed the remarkable fact that mere familiarity with things is able to produce a feeling of their rationality. The empiricist school has been so much struck by this circumstance as to have laid it down that the feeling of rationality and the feeling of familiarity are one and the same thing, and that no other kind of rationality than this exists. The daily contemplation of phenomena juxtaposed in a certain order begets an acceptance of their connection, as absolute as the repose engendered by theoretic insight into their coherence. To explain a thing is to pass easily back to its antecedents; to know it is easily to foresee its consequents. Custom, which lets us do both, is thus the source of whatever rationality the thing may gain in our thought. In the broad sense in which rationality was defined at the outset of this essay, it is perfectly apparent that custom must be one of its factors. We said that any perfectly fluent and easy thought was devoid of the sentiment of irrationality. Inasmuch then as custom acquaints us with all the relations of a thing, it teaches us to pass fluently from that thing to others, and pro tanto tinges it with the rational character. Now, there is one particular relation of greater practical importance than all

> the rest, —I mean the relation of a thing to its future consequences. So long as an object is unusual, our expectations are baffled; they are fully determined as soon as it becomes familiar. I therefore propose this as the first practical requisite which a philosophic conception must satisfy: It must, in a general way at least, banish uncertainty from the future.

11. Preparing the mind for understanding— "method"

We all have standard ways of preparing for rational deliberation. E.g., by listing Pros and Cons relevant to a given decision; or making sure that we can take a breath and deliberate in a calm (so to speak: "rational") way.

Or following a prescribed route for understanding, as in René Descartes' *Rules for the Regulation of the Mind*[31] (written by 1628 but not published until 1701).

From Rule III of the above:

> **By intuition I mean, not the wavering assurance of the senses, or the deceitful judgment of a misconstructed imagination, but a conception, formed by unclouded mental attention, so easy and distinct as to leave no room for doubt in regard to the thing we are understanding.** It comes to the same thing if we say: It is an indubitable conception formed by an unclouded mental mind; one that originates solely from the light of reason, and is more certain even than deduction, because it is simpler (though, as we have previously noted, deduction, too, cannot go wrong if it is a human being that performs it). Thus, anybody can see by *mental intuition* that he himself exists, that he thinks, that a triangle is bounded by just three lines, and a globe by a single surface, and so on; there are far more of such truths than most people observe, because they disdain to turn their mind to such easy topics. Some people may perhaps be troubled by this new use of the word intuition, and of other words that I shall later on be obliged to shift away from their common meaning. So I give at this point the general

[31] `https://canvas.harvard.edu/courses/108721/files?preview=15977868`

warning that I am not in the least thinking of the usage of particular words that has prevailed in the Schools in modern times, since it would be most difficult to use the same terms while holding quite different views; I take into account only what a given word means in Latin, in order that, whenever there are no proper words for what I mean, I may transfer to that meaning the words that seem to me most suitable. The evidentness and certainty of intuition is, moreover, necessary not only in forming propositions but also for any inferences. For example, take the inference that 2 and 2 come to the same as 3 and 1; intuition must show us not only that 2 and 2 make 4, and that 3 and 1 also make 4, but furthermore that the above third proposition is a necessary conclusion from these two. This may raise a doubt as to our reason for having added another mode of knowledge, besides intuition, in this Rule—namely, knowledge by deduction. (By this term I mean any necessary conclusion from other things known with certainty.) We had to do this because many things are known although not self-evident, so long as they are deduced from principles known to be true by a continuous and uninterrupted movement of thought, with clear intuition of each point.

. . .

From this we may gather that when propositions are direct conclusions from first principles, they may be said to be known by intuition or by deduction, according to different ways of looking at them; but first principles themselves may be said to be known only by intuition; and remote conclusions, on the other hand, only by deduction.

12. Empirical & Rational—Analytic & Synthetic

From Kant's essay: Prolegomena to Any Future Metaphysics[32]

[32] https://www.gutenberg.org/files/52821/52821-h/52821-h.htm

- **Of the Distinction between Analytical and Synthetical Judgments in general.**

> Analytical judgments express nothing in the predicate but what has been already actually thought in the concept of the subject, though not so distinctly or with the same (full) consciousness. When I say: **All bodies are extended**[33]**,** I have not amplified in the least my concept of body, but have only analysed it, as *extension was really thought to belong to that concept before the judgment was made*, though it was not expressed; this judgment is therefore *analytical*. On the contrary, this judgment, **All bodies have weight,** contains in its predicate something not actually thought in the general concept of the body; it amplifies my knowledge by adding something to my concept, and must therefore be called *synthetical*.

- **The Common Principle of all Analytical Judgments is the Law of Contradiction.**

> All analytical judgments depend wholly on the law of Contradiction, and are in their nature a priori cognitions, whether the concepts that supply them with matter be empirical or not. For the predicate of an affirmative analytical judgment is already contained in the concept of the subject, of which it cannot be denied without contradiction.
> In the same way its opposite is necessarily denied of the subject in an analytical, but negative, judgment, by the same law of contradiction. Such is the nature of the judgments: all bodies are extended, and no bodies are unextended (i.e., simple). For this very reason all analytical judgments are a priori even when the concepts are empirical, as, for example,
> Gold is a yellow metal; for to know this I require no experience beyond my concept of gold as a yellow metal: it is, in fact, the very concept, and I need only analyse it, without looking beyond it elsewhere.

[33] *extended* = 'takes up some volume of space'

- **Synthetical Judgments require a different Principle from the Law of Contradiction.**

> There are synthetical a posteriori judgments of empirical origin; but there are also others which are proved to be certain a priori, and which spring from pure Understanding and Reason. Yet they both agree in this, that they cannot possibly spring from the principle of analysis, viz., the law of contradiction, alone; they require a quite different principle, though, from whatever they may be deduced, they must be subject to the law of contradiction, which must never be violated, even though everything cannot be deduced from it. I shall first classify synthetical judgments.

1. Empirical Judgments are always synthetical...
2. Mathematical Judgments are all synthetical...

> It might at first be thought that the proposition
>
> $$7 + 5 = 12$$
>
> is a mere analytical judgment, following from the concept of the sum of seven and five, according to the law of contradiction. But on closer examination it appears that the concept of the sum of $7 + 5$ contains merely their union in a single number, without its being at all thought what the particular number is that unites them. The concept of twelve is by no means thought by merely thinking of the combination of seven and five; and analyse this possible sum as we may, we shall not discover twelve in the concept. We must go beyond these concepts, by calling to our aid some concrete image (Anschauung), i.e., either our five fingers, or five points... and we must add successively the units of the five, given in some concrete image (Anschauung), to the concept of seven. Hence our *concept* is really *amplified* by the proposition $7 + 5 = 12$, and we add to the first a second, not thought in it. Arithmetical judgments are therefore synthetical, and the more plainly according as we take larger numbers; for in such cases it is clear that, however closely we analyse

> our concepts without calling visual images (Anschauung) to our aid, we can never find the sum by such mere dissection.
> All principles of geometry are no less analytical...

Part 4. **Mathematics and Rational Thought**

13. Attitudes towards the axiomatic substructure of mathematics

Recall the overview of different *axiomatic takes* on geometry and their evolution as in our first session *Organizing Rational Thought*[34]. The concept of "axiom," as the moderns understand it, is hardly present in Euclid. His "common notions" play the role of axioms, and also sort-of axioms lurk in his definitions (e.g., *a line is breathless length*).

As we saw in the session of September 7, Hilbert's rewriting of Euclid's *Elements* has quite different style:

> Let us consider three distinct systems of things. The things composing the first system, we will call **points** and designate them by the letters A, B, C,. . . ; those of the second, we will call **straight lines** and designate them by the letters a, b, c,... The points are called the **elements of linear geometry**; the points and straight lines, **the elements of plane geometry**...

Hilbert allows himself these undefined terms: *point, line, plane, lie, between,* and *congruence.*

For a slight twist on this, now consider Hilbert's reflections on this, as in the quotations in our readings for today[35]:

> Geometry is the science that deals with the properties of space. It differs essentially from pure mathematical domains such as the theory of numbers, algebra, or the theory of functions. The results of the latter are obtained

[34]`https://canvas.harvard.edu/courses/108721/files?preview=15461024`

[35]Leo Corry's The Origins of Eternal Truth in Modern Mathematics: Hilbert to Bourbaki and Beyond, in *Science in Context*:
`https://canvas.harvard.edu/courses/108721/files?preview=16197759`

> through pure thinking ... The situation is completely different in the case of geometry. I can never penetrate the properties of space by pure reflection, much as I can never recognize the basic laws of mechanics, the law of gravitation or any other physical law in this way. Space is not a product of my reflections. Rather, it is given to me through the senses.
>
> . . .
>
> Among the appearances or facts of experience manifest to us in the observation of nature, there is a peculiar type, namely, those facts concerning the outer shape of things. Geometry deals with these facts... Geometry is a science whose essentials are developed to such a degree, that all its facts can already be logically deduced from earlier ones. Much different is the case with the theory of electricity or with optics, in which still many new facts are being discovered. Nevertheless, with regards to its origins, geometry is a natural science.

Question: What do we—and how should we—think about Hilbert's comment:

"*...with regards to its origins, geometry is a natural science*" ?

14. Indirect argument in Mathematics:

Indirect argument is often framed as "proof by contradiction." You're given an *indirect argument* if when you want to prove something (call it **A**) you prove that its negation ($\vdash$ **A**) cannot be true—i.e., leads to a contradiction. Ergo[36]: **A**.

A favorite example: prove **A** : that $\sqrt{2}$ is irrational; i.e., that the square root of 2 cannot be expressed as a ratio of whole numbers. You 'assert'

$\vdash$ **A** : 2 can be expressed as $\dfrac{a^2}{b^2}$ (with a and b relatively prime)

and note that

$$2b^2 = a^2$$

[36] following the flavor of Leibniz's notion of "compossibility"

can't be true since looking from left to right you see that a has to be even, i.e., of the form $2 \cdot c$ for some whole number c; so

$$2b^2 = 4c^2$$

and then, given that, looking from right to left you see that b has also to be even, so a and b are *not* relatively prime. I.e., you've proven $\vdash\vdash$ **A** to be true, establishing **A**.

Question: Are there there ways of proving, in effect, the same thing without explicitly drawing a contradiction?

Question: Same question for the proof that there are infinitely many prime numbers.

15. How does Mathematics answer questions beginning with the word "Why"?

The first 7 pages of my notes *On the word "Because" in mathematics and elsewhere*[37]—which I hope you've read—is meant to be an invitation to discuss the puzzling question: what do we expect for an answer when we ask about a particular mathematical fact: "*why is this true?*"?

For example, is there a satisfactory response to the question:

Why is $7 + 5 = 12$ true?

a response, that is, that is more than simply verifying *that* it is true?

Ditto with the Pythagorean Theorem.

16. Semantics versus syntax in mathematics

Hilbert's view of geometry—as illustrated by the quotations that we discussed in Section **??** above—has two phases: geometry begins its life, so to speak as a *natural science*; and then, via his *Grundlagen*, geometry is endowed with a formal structural language—"things" merely labeled *points, lines* etc. with specified combinatorial relations —for the expression and verification of arguments. This type of shift is the MO of mathematics.

[37] `https://canvas.harvard.edu/courses/108721/files?preview=15865181`

There is an interesting (and very powerful) resource in current pure mathematical thought called *Model Theory* that makes that type of shift, but with a vengeance.

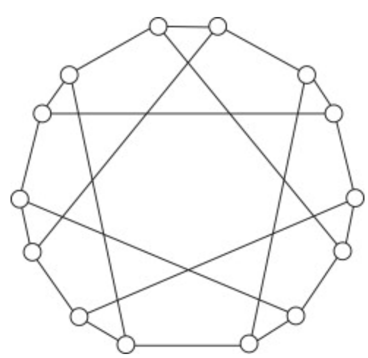
To study some specific theory comprising a constellation of mathematical concepts *Model Theory* offers a syntactical frame encoded in a crisp language for which that 'specific theory' is simply one model (perhaps of many models) that nest perfectly well within that framework.

Model Theory allows you to turn your attention from the *semantics* of that particular theory to the engulfing *syntax* of the Model Theory, with respect to which that particular theory is one of possibly many models. It provides a format for doing mathematics within an explicitly shaped 'Language' (in the style of 'universal algebra')—where the 'models' will be *sets with extra structure*—and where its *sentences* interpreted in any 'model' have truth-values that conform to the rules of first-order logic.

Here is an example of an 'opening move' of Model Theory that effects a revealing disarticulation of semantics from substance.

If you are a graph theorist it is perfectly reasonable to formulate *graph theory* as follows:

Define a **graph** to be given by a set V of vertices and a set E of edges, each edge attaching two distinct vertices and you might also insist that no two vertices are attached by more than one edge. Or you might give a more topological account of this structure.



In any event, your formulation begins with a set and then some structure is imposed on it.

Model Theory, reverses this. It begins by offering an explicitly shaped language in which first-order logic is incorporated. In the case of our example of graph theory, the language simply be:

- a symbol $\mathcal{E}$ labeled as a *symmetric binary relation* (but not reflexive)
- in connection with which we label as *true* sentences:

$$\forall x, y (x\mathcal{E}y \leftrightarrow y\mathcal{E}x)$$

 and

$$x\mathcal{E}y \implies x \neq y.$$

An 'interpretation' of this language—or synonymously, a 'model' for this would be a 'representation' of this language in (some version of) Set Theory.

That is, a 'model' would be a set V endowed with (such) a binary relation E for which the labeled-as-true sentences are... in fact true; i.e., such a model is simply a graph, where the set of vertices is the set V and the set of edges is given by the binary relation E: there's an edge linking two vertices $v, w \in V$ if and only if xEy (and hence also yEx).

All this is meant to get us to discuss the question:

Question: What is a model?