# Review

## Continuity

Use polar coordinates to test continuity

1. $f(x,y) = \frac{x^5}{x^3y+y^4+x^4}$ is continuous at $(0,0)$
2. $f(x,y) = \frac{x^3}{x^3y+y^4+x^4}$ is **NOT** continuous at $(0,0)$

## Partial derivative

1. Clairaut's theorem $f_{xy} = f_{yx}$

2. Geometric meaning

$$f_x(x,y) = D_{\langle 1,0\rangle} f(x,y)$$
$$f_y(x,y) = D_{\langle 0,1\rangle} f(x,y)$$

## PDE

1. Wave equation $u_{tt} = u_{xx}$
2. Heat equation $u_t = u_{xx}$
3. Laplace equation $u_{xx} + u_{yy} = 0$
4. Transport equation $u_t = u_x$
5. Burgers equation $u_t + uu_x = u_{xx}$

## Gradient

| | |
|---|---|
| Chain rule | $\frac{d}{dt} f(\vec{r}(t)) = \boxed{\phantom{xxxx}} \cdot \vec{r}\,'(t)$ |
| Directional derivative | $D_{\langle 1,1\rangle/\sqrt{2}} f(1,2) = \nabla f(1,2) \cdot \boxed{\phantom{xxxx}}$ |
| Linearization of $f(x,y)$ at $(1,2)$ | $L(x,y) = \boxed{\phantom{xxxx}} + \nabla f(1,2) \cdot \langle x-1, y-2\rangle$ |
| Equation of tangent line at $(1,2)$ | $\nabla f(1,2) \cdot \langle x-1, y-2\rangle = \boxed{\phantom{xxxx}}$ |

1. Chain rule $f'(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$.
2. Directional derivative $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$
3. The gradient is perpendicular to the level curves (or level surfaces)(contours)This is used to find tangent lines to the level curves or tangent planes to level surfaces
4. The gradient points toward the direction of greatest increase of the function
5. Fact 3 is used to find tangent lines to the level curves or tangent planes to level surfaces. Trick: Tangent plane to the graph of $z = f(x,y)$: use a new function $g(x,y,z) = z - f(x,y)$, the graph is also the level surface of $g(x,y,z) = 0$.

# Linear approximation

1. **Key properties of linear approximation:** The linear approximation of a function $f(x, y)$ at a point $(a, b)$ is a **linear function** $L(x, y)$ whose value and partial derivatives at $(a, b)$ are the same as those of $f(x, y)$:

$$L(a, b) = f(a, b)$$
$$L_x(a, b) = f_x(a, b)$$
$$L_y(a, b) = f_y(a, b)$$

The graph $z = L(x, y)$ is the tangent plane to the graph $z = f(x, y)$ at the point $(a, b, f(a, b))$.

The formula for linear approximation is

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).}$$

or

$$\boxed{L(x, y) = f(a, b) + \nabla f(a, b) \cdot (x - a, y - b).}$$

2. **Two methods to get tangent plane**

   - If surface is given implicitly by the level sets $F(x, y, z) = C$, the gradient of $F$ at the point $(x_0, y_0, z_0)$ gives the normal vector to the tangent plane.

$$\vec{N} = \nabla F(x_0, y_0, z_0) = (a, b, c)$$

   - If the surface is given by parametrization $\vec{r}(u, v)$, then calculate two tangent vectors $\vec{r}_u(u_0, v_0)$, $\vec{r}_v(u_0, v_0)$ and take cross product.

$$\vec{N} = \vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0) = (a, b, c)$$

# Chain Rule

- The formula for the Chain Rule is

$$\boxed{f'(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).}$$

- **How to do implicit differentiation:** If $y$ is given *implicitly* as a function of $x$ by an equation $f(x, y) = c$, we can solve for $dy/dx$ by using

$$\boxed{\tfrac{dy}{dx} = -\tfrac{f_x}{f_y}}$$

or by differentiating partially with respect to $x$ directly and then manipulating.

## Extrema problem

1. **Local extrema: first derivative test and second derivative test**
   - $\nabla f = \vec{0}$
   - When $D = 0$, the second derivative test is not conclusive.
   - When $D > 0$, local max or min. Think about $f(x, y) = x^2 + y^2$, it has a local min.
   - When $D < 0$, saddle.
2. **Lagrange method** Optimization with constraint. Geometric meaning: the two level curves are tangent.
3. **Global extrema**
   - Bolzano theorem: closed and bounded domain.
   - When the domain is not closed or bounded, we need to consider the behaviour near the boundary and at infinity.
   - Two steps to find the glocal extrema: firstly find the critical points as local extrema, secondly use the Lagrange method at the boundary

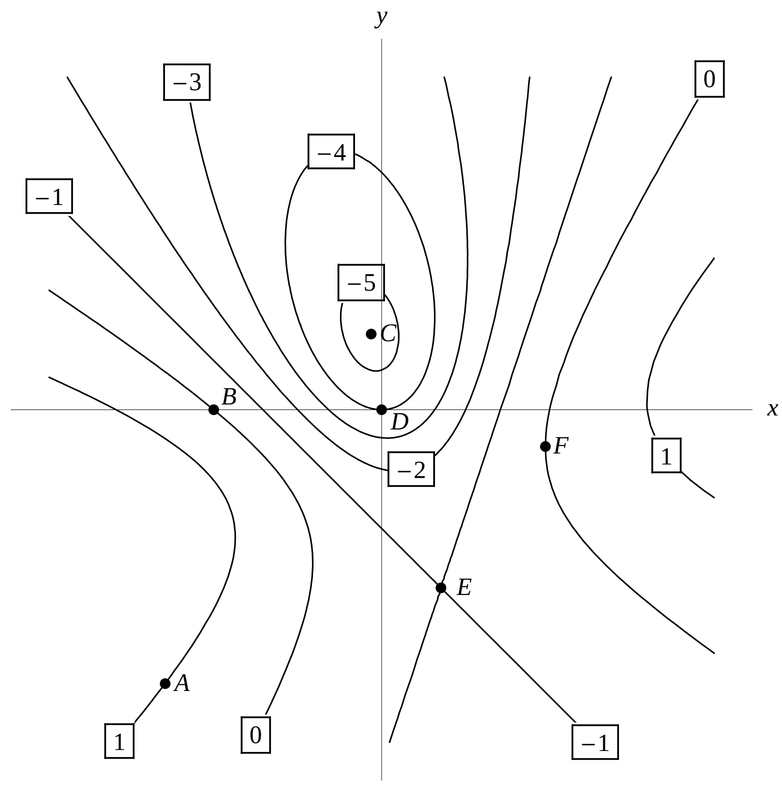
## Integration

1. Fubini's theorem: we can switch the order of the integration. Be careful about the bounds
2. Polar integral. Don't forget the distorsion factor for the area element $rdrd\theta$
3. We have two new integration techniques
   - Change between Type I and Type II, draw the picture
   - Change to polar
4. Surface area

# Exercise

## Contour map

Here is the level set diagram (contour map) of a function $f(x, y)$. The value of $f$ on each level set is indicated. Two of the six labeled points are critical points of $f$.




(a) Which two points are critical points of $f$?

$A$ $B$ $C$ $D$ $E$ $F$

Classify each critical point as a local minimum, local maximum, or saddle point.

- Point _______ is a ________________________________.
- Point _______ is a ________________________________.

(b) Two of the following four points have the property that $\frac{\partial f}{\partial x}$ is 0 at the point. Which two?

$B$ $D$ $E$ $F$

(c) Decide whether the following directional derivatives are greater than 0, less than 0, or equal to 0. Circle the appropriate phrase.

(a) $D_{\vec{u}} f(A)$, where $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

greater than 0 less than 0 equal to 0

(b) $D_{\vec{u}}f(C)$, where $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

greater than 0 less than 0 equal to 0

(c) $D_{\vec{u}}f(D)$, where $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

greater than 0 less than 0 equal to 0

## Integration

The real trouble begins when you can't integrate without switching the order of integration. Here are four examples of this. You should draw the region of integration!

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

$$\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x}\,dx\,dy$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \left(3x - x^3\right)^{10}\,dx\,dy$$

$$\int_1^{e^3} \int_{\ln(y)}^{3} \left(e^x - x\right)^5\,dx\,dy$$

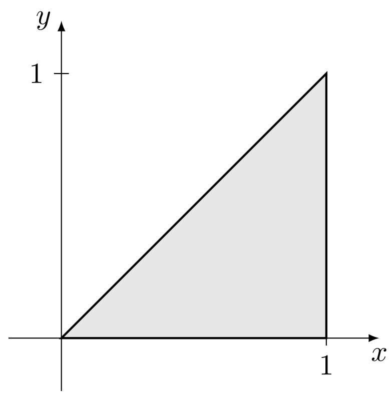
# Solutions

## Contour map

1. Critical points of $f$: $C$, $E$. Point $C$ is a local minimum, point $E$ is a saddle point.
2. $\frac{\partial f}{\partial x} = 0$ at $D$ and $E$ (and of course also $C$, but $C$ is not in the list).
3. i. less than 0; ii. equal to 0; iii. less than 0

## Integration

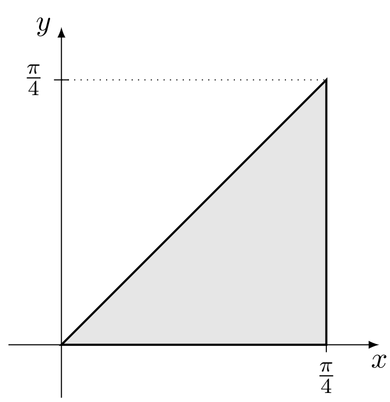
1. The region of integration is




Thus the integral can be re-written as

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 x e^{x^2}\, dx = \frac{1}{2}(e-1)\,.$$

2. The region of integration is

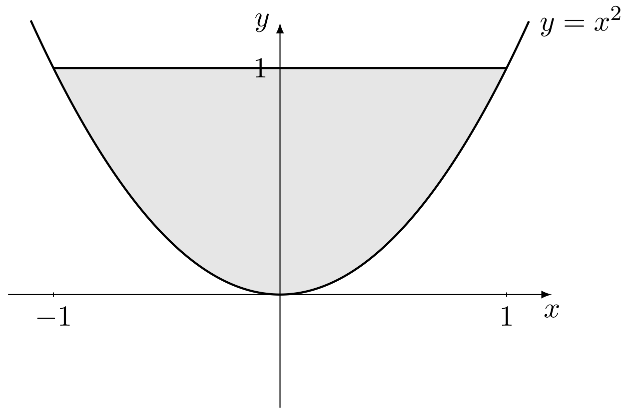



Thus the integral can be re-written as

$$\int_0^{\pi/2} \int_0^x \frac{\sin(x)}{x}\, dy\, dx = \int_0^1 \sin(x)\, dx = 1.$$

Note that the function $\frac{\sin(x)}{x}$ is undefined at $x = 0$, which in this region is simply the origin. But since $\frac{\sin(x)}{x} \to 1$ as $x \to 0$, we can extend this integrand to be continuous at this point. Let's assume that this is what we've done.

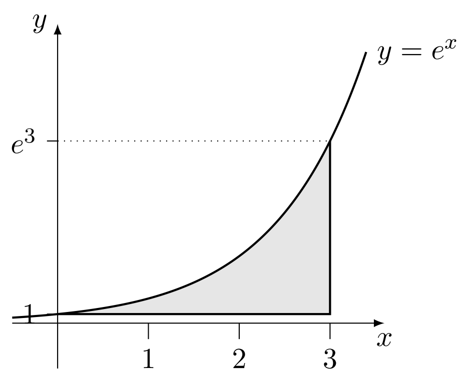
3. The region of integration is




Thus we can write the integral as

$$\int_{-1}^{1}\int_{x^2}^{1}\left(3x - x^3\right)^{10}\,dy\,dx = \int_{-1}^{1}\left(3x - x^3\right)^{10}(1 - x^2)\,dx = \frac{2^{12}}{33}.$$

4. The region of integration is




Thus we can write the integral as

$$\int_0^3\int_1^{e^x}(e^x - x)^5\,dy\,dx = \int_0^3 (e^x - x)^5\,(e^x - 1)\,dx = \frac{1}{6}\left(e^3 - 3\right)^6 - \frac{1}{6}.$$