SECOND HOURLY in 21 TWEETS Math21a Fall 2016

Each tweet **140** characters or less.

Clairaut: $f_{xy} = f_{yx}$ Proof:
$f_{xy} \sim h^2[(f(x+h,y+h) - f(x,y+h)) - (f(x+h,y) - f(x,y))]$
$f_{yx} \sim h^2[(f(x+h,y+h) - f(x+h,y)) - (f(x,y+h) - f(x,y))]$

ODE: equation for function f involving derivatives
PDE: equation for function f involving partial derivatives. Example: $f_x = f_{yy} f + f^2$

Heat equation: $u_t = u_{xx}$
Wave equation: $u_{tt} = u_{xx}$
Laplace equation $u_{xx} + y_{yy} = 0$
Transport equation $u_t = u_x$
Burgers equation $u_t + uu_x = u_{xx}$

Gradients are perpendicular to level sets.
Proof: $\vec{r}(t)$ on $f = c$ satisfies
$0 = d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
showing that ∇f is orthogonal

Directional derivative is max in gradient direction. Proof:
$D_{\vec{v}} f = |\nabla f \cdot \vec{v}| = |\nabla f| \cos(a)$
For $\vec{w} = \nabla f / |\nabla f|$,
$D_{\vec{w}} f = |\nabla f|$

Second derivative test: discriminant
$D = f_{xx} f_{yy} - f_{xy}^2$ and $A = f_{xx}$ determine:
$D > 0, A > 0 \Rightarrow$ min, $D > 0, A < 0 \Rightarrow$ max,
$D < 0 \Rightarrow$ saddle, $D = 0$ not know.

Lagrange:
$f_x = L g_x$
$f_y = L g_y$
$g(x,y) = c$
Proof: $\nabla(f)$ and $\nabla(g)$ are parallel. Else moving on $g = c$ crosses level curves.

Chain rule $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. Implicit differentiation $f(x, g(x)) = 0$ implies $g'(x) = -f_x/f_y$.

Tangent plane at P : find $\nabla f = \langle a, b, c\rangle$ and plane $ax + by + cz = d$, (get d by plugging in P).
$\nabla f = \langle a, b\rangle$ gives tangent line $ax + by = d$.

Estimate $f(3.001, 4.9999)$ by computing the gradient $\langle a, b\rangle$ of f at $(3,5)$ and get
$L(3.001, 4.9999) = f(3,5) + a \cdot 0.001 - b \cdot 0.0001$.

Type I = "bottom to top" integration on $[a,b]$ on x-axis
Type II = "left to right" integration on $[c,d]$ on y-axis

Double integral $\int\int_R f(x,y)\, dxdy$ interpretation:
signed volume under the graph of f. It is volume if $f \geq 0$.

Fubini: for rectangular regions only:
$\int_a^b \int_c^d f(x,y) dxdy = \int_c^d \int_a^b f(x,y) dydx$

Surface area of a parametrized surface $r(u,v)$, defined for a region R is $\int\int_R |\vec{r}_u \times \vec{r}_v| dudv$.

Polar integration:
include an integration factor r.
Proof: $\vec{r}(s,t) = \langle r\cos(t), r\sin(t), 0\rangle$, $|\vec{r}_r \times \vec{r}_t| = r$.

By parts: $\int u dv = uv - \int v du$
Proof: integrate $uv' + vu' = (uv)'$
Example: $\int x\cos(x) dx = x\sin(x) - \int 1 \cdot \sin(x) dx = x\sin(x) + \cos(x) + C$

Substitution: Example:
$\int x^4 \exp(x^5) dx$
$u = x^5, du = 5x^4 dx$ $\int \exp(u)/5 du = \exp(x^5)/5 + C$

Tips for 2D integrals: make a picture, consider other coordinates and change order of integration.

Helpful identities:
$\cos^2(t) + \sin^2(t) = 1$
$\cos^2(t) = (1 + \cos(2t))/2$
$\sin^2(t) = (1 - \cos(2t))/2$
$\sin(t)\cos(t) = \sin(2t)/2$

$\int x^n dx = x^{n+1}/(n+1)$
$\int \exp(ax) dx = \exp(ax)/a$
$\int \cos(ax) = \sin(ax)/a$
$\int \sin(ax) = -\cos(ax)/a$
$\int dx/x = \log(x)$
$\int dx/(1+x^2) = \arctan(x)$

Know integration by parts, substitution, basic integrals!

Oliver Knill, October 30, 2016.

CHECKLIST SECOND HOURLY Math21a Fall 2016

Partial Differential equations

☐ $u_t = u_{xx}$ heat equation
☐ $u_{tt} - u_{xx} = 0$ wave equation
☐ $u_t - u_x = 0$ transport equation
☐ $u_{xx} + u_{yy} = 0$ Laplace equation
☐ $u_t + uu_x = u_{xx}$ Burgers equation
☐ $u_{xy} = u_{yx}$ Clairaut theorem

Linearization

☐ $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ linear approximation
☐ use $L(x,y)$ to estimate $f(x,y)$ near $f(x_0,y_0)$. The result is $f(x_0,y_0)+a(x-x_0)+b(y-y_0)$
☐ tangent line: $ax+by=d$ with $a=f_x(x_0,y_0), b=f_y(x_0,y_0)$, $d=ax_0+by_0$
☐ tangent plane: $ax+by+cz=d$ with $a=f_x, b=f_y, c=f_z$, $d=ax_0+by_0+cz_0$
☐ estimate $f(x,y)$ by $L(x,y)$ near (x_0,y_0)
☐ estimate $f(x,y,z)$ by $L(x,y,z)$ near (x_0,y_0,z_0)

Gradient and Tangent spaces

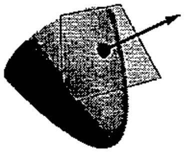
☐ $\nabla f(x,y) = \langle f_x, f_y \rangle$, gradient in two dimensions
☐ $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$, gradient in three dimensions
☐ $\nabla f(x_0,y_0,z_0)$ is orthogonal to the level surface $f(x,y,z)=c$ containing (x_0,y_0,z_0)
☐ $\nabla f = \langle a,b,c \rangle$ defines tangent plane $ax+by+cz=d$, where $d=ax_0+by_0+cz_0$

Chain rule

☐ $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ chain rule
☐ $f(x,y)=c$ defines $y=g(x)$, and $g_x = -f_x/f_y$ implicit differentiation
☐ $f(x,y,z)=c$ defines $z=g(x,y)$, and $g_x = -f_x/f_z$ implicit differentiation

Directional Derivative

☐ $D_{\vec{v}} f = \nabla f \cdot v$ directional derivative, where $\vec{v}$ is a unit vector
☐ directional derivative is maximal in the $\vec{v} = \nabla f/|\nabla f|$ direction
☐ $f(x,y)$ increases in the $\nabla f/|\nabla f|$ direction
☐ $\nabla f/|\nabla f|$ direction of steepest ascent
☐ $-\nabla f/|\nabla f|$ direction of steepest descent
☐ partial derivatives are special directional derivatives
☐ the length of $|\nabla f|$ is the steepness in the $\nabla f/|\nabla f|$ direction



Extrema

☐ $\nabla f(x,y) = \langle 0,0 \rangle$, critical point or stationary point
☐ $D = f_{xx}f_{yy} - f_{xy}^2$ discriminant
☐ $f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0): local maximum
☐ $f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0): local minimum
☐ test: $D>0, f_{xx}<0$ local max, $D>0, f_{xx}>0$ local min, $D<0$ saddle

Lagrange equations

☐ $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$

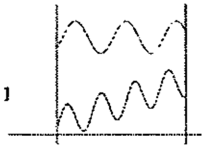
☐ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z), g(x,y,z) = c$

☐ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), g(x,y,z) = c, h(x,y,z) = d$
☐ Strategy: eliminate λ or λ, μ first.

Global Extrema

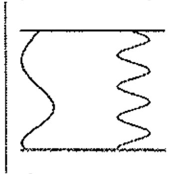
☐ Extremal value theorem of Bolzano: continuous f on closed bounded region attains max.
☐ (x_0,y_0) is a global maximum of $f(x,y)$, if $f(x,y) \leq f(x_0,y_0)$ for all (x,y).
☐ To find it, compare all local extrema inside the domain as well as extrema on the boundary. For unbounded regions, investigate x,y go to infinity.

Double Integrals



☐ $\int\int_R f(x,y)\, dydx$ double integral
☐ $\int\int_R f(x,y)\, dxdy$ is the signed volume bound by graph(f) and xy-plane
☐ $\int_a^b \int_c^d f(x,y)\, dydx$ integral over rectangle
☐ $\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx$ type I region
☐ $\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dxdy$ type II region
☐ $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$ Fubini (for rectangles!)
☐ $\int\int_R 1\, dxdy$ area of region R=Volume under constant function $f=1$

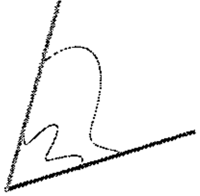
Integrals in polar Coordinates



☐ $\int\int_R f(r\cos(\theta), r\sin(\theta)) \boxed{r}\, drd\theta$ polar coordinates

Surface area

☐ $\int\int_R |\vec{r}_u \times \vec{r}_v|\, dudv$, where $\vec{r}(u,v)$ is a parametrization over R.
☐ $|\vec{r}_\theta \times \vec{r}_z| = g(z)\sqrt{1+g'(z)^2}$ if $\vec{r}(\theta,z) = \langle g(z)\cos(\theta), g(z)\sin(\theta), z \rangle$.
☐ $\int\int_R \sqrt{1+f_x^2+f_y^2}\, dxdy$ if $\vec{r}(x,y) = \langle x,y,f(x,y) \rangle$.



O. Knill, October 30, 2016