NOTES ON HIGHER DIMENSIONAL DERIVATIVES

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1. Single-variable recap

Let $f(x)$ be a function of one variable and $I = [a, b] \subset \mathbb{R}$ an interval.

Definition 1.1. The *integral of f over I* is

$$\int_I f(x)dx = \lim_{I=\cup I_j} \sum_j \text{length}(I_j) f(p_j) \tag{1}$$

where I_j are intervals *partitioning* I and p_j is a point in I_j.

Remark 1.2. There are at least two ways of interpreting this definition: The integral of f over I is the...

(1) ... "weighted length" of I.
(2) ... "signed area" under the graph of f.

2. More variables

We want a notion of integration over higher dimensional spaces which generalizes the single variable integral. Let's try to just replace the 1-dimensional words in Definition 1.1 with 2-dimensional words for starters.

Let $f(x, y)$ be a continuous function of two variables and $R \subset \mathbb{R}^2$ a two dimensional region in the plane.

Definition 2.1. The *(double) integral of f over R* is

$$\int_R f(x, y)dA = \lim_{R=\cup R_j} \sum_j \text{Area}(R_j) f(p_j) \tag{2}$$

where R_j are rectangles partitioning R and p_j is a point in R_j.

Remark 2.2. There are at least two ways of interpreting this definition: The integral of f over R is the...

(1) ... "weighted area" of R.
(2) ... "signed volume" of the graph of f.

Theorem 2.3 (Fubini's Theorem)**.** *If $R = [a, b] \times [c, d]$ is a rectangle, then $\int_R f dA = \int_c^d \int_a^b f dx dy = \int_a^b \int_c^d f dy dx$.*

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Proof. Partition the rectangular region $R = [a, b] \times [c, d]$ into many small rows R_i with heights h_i and columns C_j with widths w_j. Let (x_j, y_i) be a point in $R_i \cap C_j$. Then

$$\begin{aligned}
\int_R f(x,y)dA &= \lim \sum_{i,j} \text{Area}(R_i \cap C_j) f(x_j, y_i) \\
&= \lim \sum_{i,j} h_i w_j f(x_j, y_i) \\
&= \lim \sum_i h_i (\sum_j w_j f(x_j, y_i)) \\
&= \lim \sum_i h_i (\int_a^b f(x, y_j)dx) \\
&= \lim \int_c^d (\int_a^b f(x,y)dx)dy.
\end{aligned}$$

However, we could have summed over the columns *before* the rows and obtained

$$\begin{aligned}
\int_R f(x,y)dA &= \lim \sum_{i,j} \text{Area}(R_i \cap C_j) f(x_j, y_i) \\
&= \lim \sum_{i,j} h_i w_j f(x_j, y_i) \\
&= \lim \sum_j w_j (\sum_i h_i f(x_j, y_i)) \\
&= \lim \sum_j w_j (\int_c^d f(x_i, y)dx) \\
&= \lim \int_a^b (\int_c^d f(x,y)dx)dy.
\end{aligned}$$

□

Remark 2.4. Fubini's theorem is important for two reasons:

1. It let's us compute double integrals as iterated (1-variable) integrals.
2. It tells us that the order of the iterated integrals doesn't matter.