

CHANGE OF COORDINATES

JAKE MARCINEK
HARVARD UNIVERSITY
CAMBRIDGE, MA
MARCINEK@MATH.HARVARD.EDU

1. SINGLE-VARIABLE RECAP

Let $\vec{r} : R \rightarrow S$ be a parametrization of a surface $S \subset \mathbb{R}^3$ by a region $R \subset \mathbb{R}^2$, $\langle x, y, z \rangle = \vec{r}(u, v)$.

Theorem 1.1 (Change of variables). *The area form of S is $dA = |\vec{r}_u \times \vec{r}_v| du dv$.*

Corollary 1.2 (Polar coordinates). *Parametrize by polar coordinates $\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta \rangle$. Then*

$$\vec{r}_r = \langle \cos \theta, \sin \theta \rangle, \quad (1)$$

$$\vec{r}_\theta = \langle -r \sin \theta, r \cos \theta \rangle, \text{ so (plugging in } \theta\text{'s for the } z\text{-coordinates)} \quad (2)$$

$$\vec{r}_r \times \vec{r}_\theta = \langle 0, 0, r \rangle, \text{ and} \quad (3)$$

$$|\vec{r}_\theta \times r_\varphi| = r. \quad (4)$$

Therefore, the area form is $dA = dx dy = r dr d\theta$.

Corollary 1.3 (Surface area). *If a surface is parametrized by $\vec{r}(u, v)$ for $(u, v) \in R \subset \mathbb{R}^2$, then $\text{Area}(S) = \int_S 1 dA = \int_R |\vec{r}_u \times \vec{r}_v| du dv$.*

Example 1.4 (Unit sphere). Find the area of the unit sphere, S^2 .

Parametrize S by $\vec{r}(\theta, \varphi) = \langle \cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi \rangle$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$. Then

$$\vec{r}_\theta = \langle -\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0 \rangle, \quad (5)$$

$$\vec{r}_\varphi = \langle \cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi \rangle, \text{ so} \quad (6)$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \langle -\cos \theta \sin^2 \varphi, \sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi \rangle, \text{ and} \quad (7)$$

$$|\vec{r}_\theta \times r_\varphi| = \sin \varphi. \quad (8)$$

Putting everything together,

$$\text{Area}(S^2) = \int \int_{S^2} 1 dA = \int_0^\pi \int_0^{2\pi} \sin \varphi d\theta d\varphi = 2\pi \int_0^\pi \sin \varphi d\varphi = 4\pi. \quad (9)$$