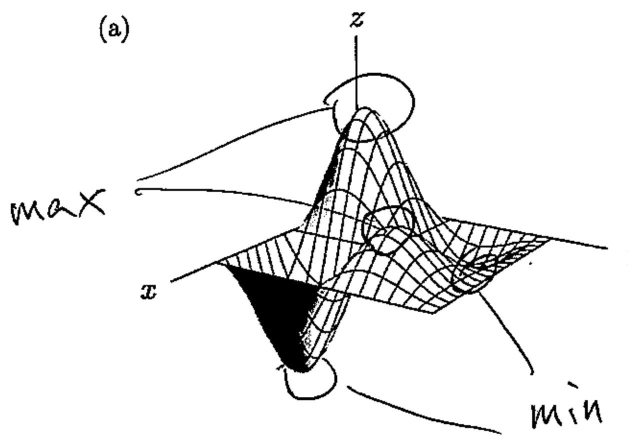
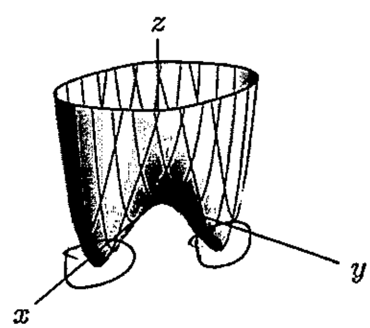
Maxima and Minima

1. In each part, you're given the graph of a function $f(x, y)$. Mark the local minima and local maxima of f on the graph.

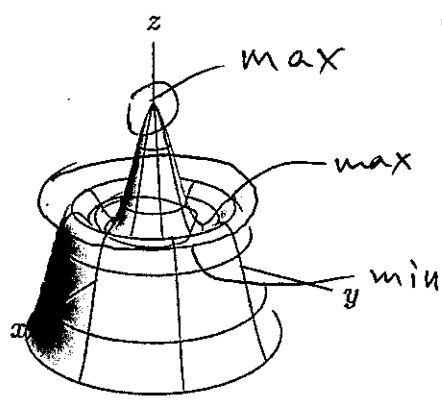
(a)



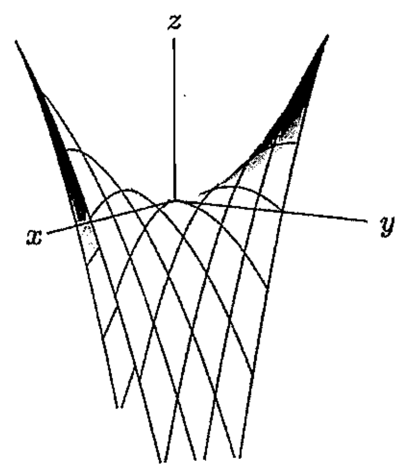
(b)



(c)



(d)



BTW saddles in (a) (b) (d)

max: concave down in every direction
min: concave up in every direction
saddle: concave down in some, up in others

2. Find the critical points of each function.

(a) $f(x,y) = x^2 + y^2 + xy$

$$0 = \nabla f(x,y) = \langle 2x + y,\ 2y + x \rangle$$

$$\begin{cases} 0 = 2x + y \\ 0 = 2y + x \end{cases} \implies y = -2x = -2(-2y) = 4y \implies y = 0$$

$$\implies x = -2y = 0$$

$$\boxed{(0,0)}$$

(b) $f(x,y) = x^4 + y^4 + 4xy$

$$0 = \nabla f = \langle 4x^3 + 4y,\ 4y^3 + 4x \rangle$$

$$\begin{cases} 0 = y + x^3 \\ 0 = x + y^3 \end{cases} \implies y = -x^3 = -(-y^3)^3 = y^9$$

$$0 = y^9 - y = y(y^8 - 1) \text{ so } y = 0, \pm 1$$

$$x = 0, \mp 1$$

$$\boxed{\therefore (0,0)\ (1,-1)\ (-1,1)}$$

3. Classify each of the critical points you found in #2 as a local minimum, local maximum, or saddle point.

$$\nabla\nabla f = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}$$

crit pts	$(0,0)$	$(1,-1)$	$(-1,1)$
$\nabla\nabla f$	$\begin{matrix} 0 & 4 \\ 4 & 0 \end{matrix}$	$\begin{matrix} 12 & 4 \\ 4 & 12 \end{matrix}$	$\begin{matrix} 12 & 4 \\ 4 & 12 \end{matrix}$
$D = \det \nabla\nabla f$	-16	128	128
f_{xx}	0	12	12
min/max/saddle	saddle	min	min

4. (a) Does $f(x,y) = x^2 + y^2$ attain an absolute minimum? An absolute maximum?

Yes (0,0) No

(b) Does $f(x,y) = x^2 + y^2$ attain an absolute minimum on the square $|x| \leq 1, |y| \leq 1$? An absolute maximum?

Yes Yes

(c) Does $f(x,y) = x^2 + y^2$ attain an absolute minimum on the square $|x| < 1, |y| < 1$? An absolute maximum?

No Yes

not closed

(d) Does $f(x,y) = x^2 + y^2$ attain an absolute minimum on $|x| \leq 1$? An absolute maximum?

Yes No not bounded

5. (a) The Extreme Value Theorem guarantees that $f(x,y) = x^2 + y^2 + xy$ (the function from #2(a)) attains an absolute minimum and an absolute maximum on $x^2 + y^2 \leq 4$. Find these absolute maximum and minimum values.

EVT domain closed and bounded implies there exist global max and min

Idea: interior + boundary = disk

global

① first check for local extrema on the interior $x^2+y^2<4$

$\nabla f = \langle 2x+y, 2y+x \rangle \Rightarrow (0,0)$

$f(0,0) = 0$

② Next check candidates on the boundary

$\vec{r}(\theta) = \langle 2\cos(\theta), 2\sin(\theta) \rangle$

$f(\vec{r}(\theta)) = 4\cos^2\theta + 4\sin^2\theta + 4\cos\theta\sin\theta = 4 + 2\sin 2\theta$

max'd at $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$, $f = 6$

min at $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, $f = 2$

(b) Does f attain an absolute minimum and an absolute maximum on $\mathbb{R}^2$? Explain how you can be sure.

Yes No

$f = \dfrac{x^2+y^2+(x+y)^2}{2}$

each term minimized at (0,0)

Take $(x,0)$ as $x \to \infty$, $f \to \infty$.

6. Does the function $f(x,y) = x^2 + 2y^2$ attain an absolute minimum on $3x + 5y = 1$? An absolute maximum?

Yes No

7. (a) Find all critical points of $f(x, y) = xy^2 - x^2 - 2y^2$ and determine whether each is a local minimum, local maximum, or saddle point.

$0 = \nabla f = \langle y^2 - 2x, 2xy - 4y \rangle$

$\begin{cases} 4y = 2xy \\ 2x = y^2 \end{cases} = y^3 \implies y(y^2-4)=0 \implies y = 0, \pm 2$

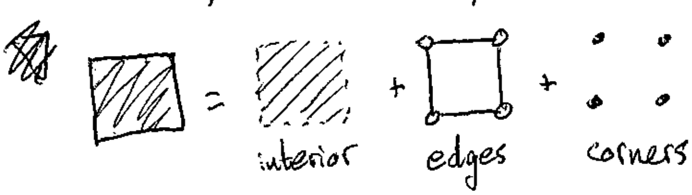
$x = 0, 2$

$\nabla\nabla f = \begin{pmatrix} -2 & 2y \\ 2y & 2x-4 \end{pmatrix}$

crit pts	(0,0)	(2,2)	(2,−2)
$\nabla\nabla f$	$\begin{smallmatrix} -2 & 0 \\ 0 & -4 \end{smallmatrix}$	$\begin{smallmatrix} -2 & 4 \\ 4 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -2 & -4 \\ -4 & 0 \end{smallmatrix}$
D	8	−16	−16
f_{xx}	−2		
min/max/saddle	max	saddle	saddle

(b) Does f attain an absolute maximum on the rectangle $0 \le x \le 2$, $0 \le y \le 4$? If so, where? If not, why not? How about an absolute minimum?

Yes, by EVT since ▨ is closed and bdd.



int: $\nabla f = 0$ does not occur on the interior so max is on the boundary

edges: $y=0 \rightsquigarrow$ max'd at $x=0$. $f(0,0)=0$

$y=4 \rightsquigarrow f = 16x - x^2 - 16$ maxed at $x=\frac{1}{2}$ $\quad f(\frac{1}{2}, 4) = 8 - \frac{1}{4} - 16 = -8\frac{1}{4} = -\frac{33}{4}$

$x=0 \rightsquigarrow$ maxed at $y=0$ $\quad f(0,0)=0$

$x=2 \rightsquigarrow$ $f(2,y) = 2y^2 - 4 - 2y^2 = -4$

$\therefore$ min $= -\frac{33}{4}$ at $(\frac{1}{2}, 4)$

max $= 0$ at $(0,0)$

(c) Does f attain an absolute maximum on the rectangle $0 < x < 2$, $0 < y < 4$? If so, where? If not, why not? How about an absolute minimum?

No No

This box is not closed. The max/min should appear on the boundary.