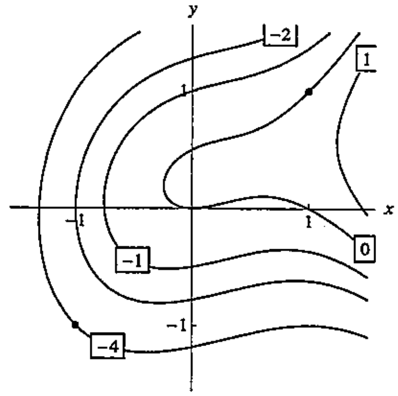
Linearization and the chain rule

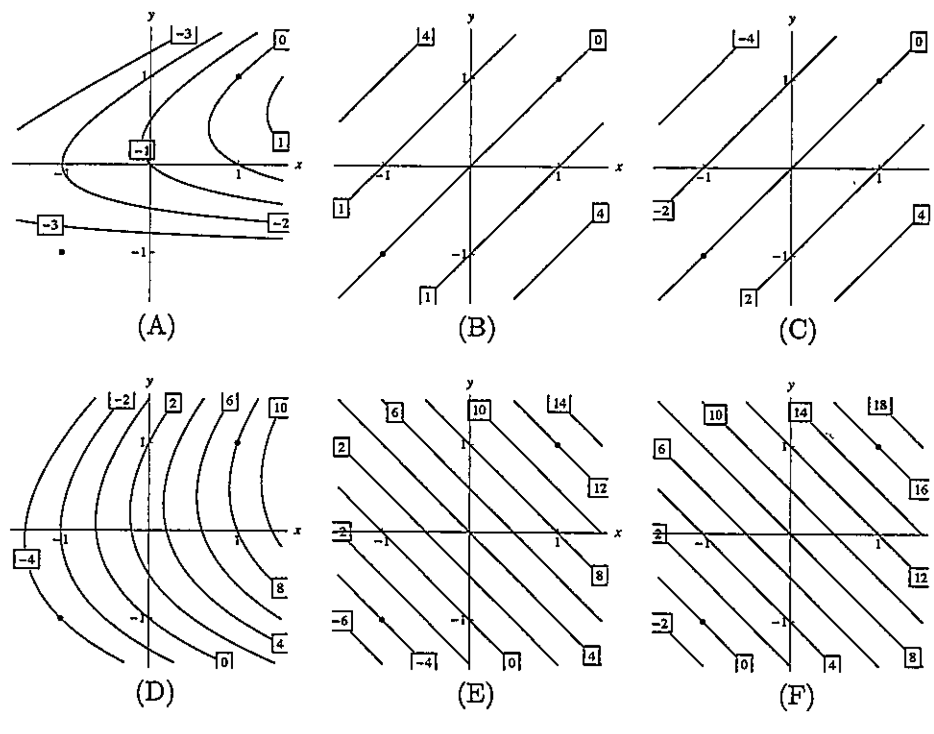
1. Suppose the mysterious function $f(x, y)$ has the following level set diagram (contour map).



The points $(1,1)$ and $(-1,-1)$ are marked with dots. Let $L_1(x, y)$ be the linearization of f at $(1,1)$ and $L_2(x, y)$ be the linearization of f at $(-1,-1)$. Which of the following is the level set diagram of $L_1(x, y)$? Which of the following is the level set diagram of $L_2(x, y)$?

$L_1 \longleftrightarrow (C)$

$L_2 \longleftrightarrow (F)$



L_f linear so level sets are lines. 1^{st} partial derivatives of f and L_f agree so a level set of L_f is a tangent line of f. Lastly, look at the direction of increase

0 level set of L_f is the tangent line of f

2. Find the tangent plane to ______ at the point ______.

- the cylinder $\{(x, y, z) \mid x^2 + y^2 = 4\}$; $(1, \sqrt{3}, 5)$
- the graph $\{z = y \sin x\}$; $(\pi/6, 2, 1)$

$L(x,y,z) = 2(x-1) + 2(y-\sqrt{3})$

Tangent plane: $L = 4$

$y \sin x - z = 0$

$L(x,y,z) = 2\frac{\sqrt{3}}{2}(x - \pi/6) + \frac{1}{2}(y-2) - (z-1)$

Tangent Plane: $L = 0$

$$L(x,y,z) = f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b))$$

- the graph $\{z = f(x,y)\}$; (a,b). $L = 0$

3. Warm-up problem: A clown is inflating a spherical balloon so that its radius at time t is $\ln(1+t)$. Find the rate at which the volume of the balloon is changing at time t. (Remember that the volume of a sphere of radius x is $\frac{4}{3}\pi x^3$.)

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left[\ln(1+t)\right]^3$$

$$R = \ln(1+t) \qquad \Longrightarrow \frac{\partial V}{\partial t} = \frac{4\pi\left[\ln(1+t)\right]^2}{1+t}$$

4. An ant is walking around on the blackboard. The temperature on the blackboard at the point (x,y) is x^4y^2. The ant's position at time t is given by the vector-valued function $\vec{r}(t) = \langle \cos t, e^t\rangle$. What is the rate of change of temperature experienced by the ant (with respect to time) at any time t?

$$T = x^4y^2 = \cos^4 t\, e^{2t}$$

$$\vec{r}(t) = \langle \cos t, e^t\rangle$$

$$\frac{\partial T}{\partial t} = 4\cos^3 t\, e^{2t} + 2\cos^4 t\, e^{2t}$$

5. Quick gradient practice: Find the gradient ∇f of the following functions f.

   (a) $f(x,y) = x^2 + y^2$. $\langle 2x, 2y\rangle$

   (b) $f(x,y,z) = x^2 + y^2 + z^2$. $\langle 2x, 2y, 2z\rangle$

   (c) $f(x,y) = xy$. $\langle y, x\rangle$

   (d) $f(x,y,z) = xyz$. $\langle yz, xz, xy\rangle$

6. A fly is flying around a room; his position at time t is $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. The temperature in the room is given by the function $f(x,y,z) = xyz$. What is the rate of change of the temperature experienced by the fly at time t?

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle \qquad x = \cos t,\ y = \sin t,\ z = t$$

$$\frac{\partial}{\partial t} f(\vec{r}(t)) = \nabla f \cdot \vec{r}_t = \langle yz, xz, xy\rangle \cdot \langle -\sin t, \cos t, 1\rangle$$

$$= \langle t\sin t, t\cos t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle$$

$$= -t\sin^2 t + t\cos^2 t + \cos t \sin t$$

(any of)
You can use 3 methods for all problems

(A) Plug in like Problem 4

(B) $\frac{\partial}{\partial t} f(\vec{r}(t)) = \nabla f(r(t)) \cdot \vec{r}_t(t)$ like Problem 6

(C) Make diagram of dependencie

7. Suppose $z = x^3 + xy + \cos y$, $x = t^2$, and $y = e^t$. Find $\frac{dz}{dt}$.

(A) $z = t^6 + t^2 e^t + \cos e^t$

$\boxed{\frac{\partial z}{\partial t} = 6t^5 + 2te^t + t^2 e^t - (\sin e^t) e^t}$

(B) $\vec{r}(t) = \langle x(t), y(t) \rangle$

$\frac{d}{dt} z(\vec{r}(t)) = \nabla z \cdot \vec{r}_t$

$= \langle 3x^2 + y, x - \sin y \rangle \cdot \langle 2t, e^t \rangle$

$= \boxed{(3t^4 + e^t) \cdot 2t + (t^2 - \sin e^t) e^t}$

(C) 

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

$= (3x+y) \cdot (2t) + (x - \sin y)(e$

$= \boxed{(3t^4 + e^t) 2t + (t^2 - \sin y) e}$

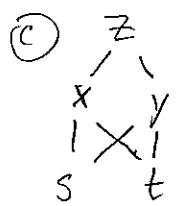
8. Suppose $u = x^2 + y^2 + z^2$, $x = s^2$, $y = \sin s$, and $z = e^s$. Find $\frac{du}{ds}$.

(B) $\frac{d}{ds} u(x(t), y(t), z(t)) = \nabla u \cdot \frac{d}{dt} \langle x(t), y(t), z(t) \rangle$

$= \underbrace{\langle 2x, 2y, 2z \rangle}_{\langle 2s^2, 2\sin s, 2e^s \rangle} \cdot \langle 2s, \cos s, e^s \rangle$

$= \boxed{4s^3 + 2\sin s \cos s + 2e^{2s}}$

9. Suppose $z = x^2 - y^2$, $x = \sin st$, and $y = te^s$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(C) 

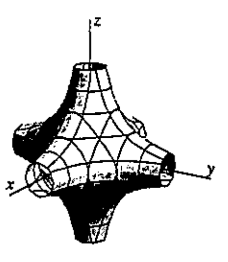
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (2x)(t \cos st) + (-2y)(te^s)$

$= \boxed{2t \sin(2st) - 2t^2 e^{2s}}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (2x)(s \sin st) + (-2y)(e^s)$

$= \boxed{2s \sin(2st) - 2te^{2s}}$

10. (Implicit differentiation.) The equation $x^2y^2 + y^2z^2 + x^2z^2 = 9$ describes the surface shown. Find $\frac{\partial z}{\partial x}$ at the point $(-1, 1, -2)$.



Apply $\frac{\partial}{\partial x}$ to

$\frac{\partial}{\partial x} \left[x^2y^2 + y^2z^2 + x^2z^2 \right] = \frac{\partial}{\partial x} 9$

$2xy^2 + 2yy_x + 2yy_x z^2 + y^2 2zz_x + 2xz^2 + x^2 2zz_x = 0$

Now set $y_x = 0$ since we are interested in how z changes as we change x

$2xy^2 + 2y^2 zz_x + 2xz^2 + x^2 2zz_x = 0$

Solve for $z_x = \frac{\partial z}{\partial x}$

$z_x = -\frac{2xy^2 + 2xz^2}{2y^2 z + 2x^2 z} = \frac{2(-1)(1)^2 + 2(-1)(-2)^2}{2(1)^2(-2) + 2(-1)^2(-2)}$

$= -\frac{-2-8}{-4-4} = \boxed{-\frac{5}{4}}$