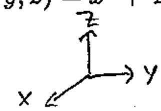
Functions, Limits, and Continuity

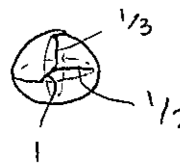
1. Describe the level sets of the following functions. What shape are they?

   (a) $f(x, y) = x^2 + 4y^2$.

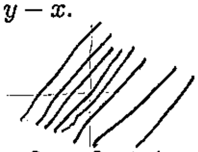
   (b) $f(x, y, z) = x^2 + 4y^2 + 9z^2$.

   & all scaled copies centered at (0,0,0)

   (c) $f(x, y) = y - x$.

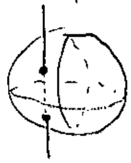
   (d) $f(x, y, z) = 2x + 3y + 4z$.

   (e) $f(x, y, z) = 4x^2 + 9y^2$.

2. Let S be the unit sphere centered at $(0, 0, 0)$. Is S the graph of a function? If so, what function?

   Is S a level set of a function? If so, what function?

   Not a function (line test)

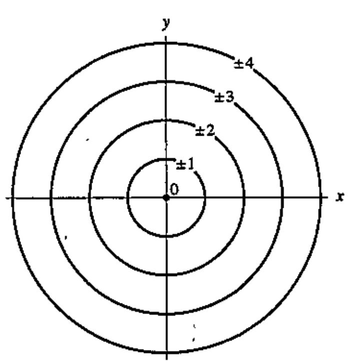
   2 intersection pts

   Yes, level set of

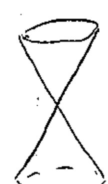
   $f(x,y,z) = x^2 + y^2 + z^2$

   Level Set $L_1 = \{ f(x,y,z) = 1 \}$.

3. Is the following picture the level set diagram (also known as contour map) of a function? If so, sketch the graph of the function.



Imagine



in 3D

$x^2+y^2=z^2$

$f(x,y) = \pm\sqrt{x^2+y^2}$

$f(r,\theta) = r$

4. Match each function with its level set diagram and its graph. (Note that each function is undefined at (0,0).)

(a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$. (Hint: What are the level sets $f(x,y)=0$, $f(x,y)=\frac{1}{2}$, and $f(x,y)=1$?)

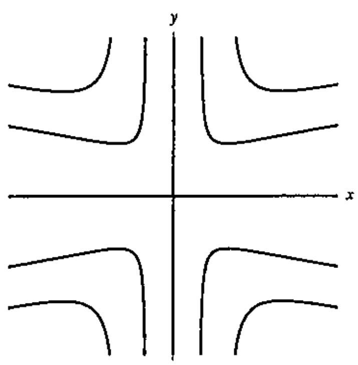
(a) (III) (B)

(b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$. (Hint: What are the level sets $f(x,y)=\frac{1}{2}$ and $f(x,y)=-\frac{1}{2}$?)

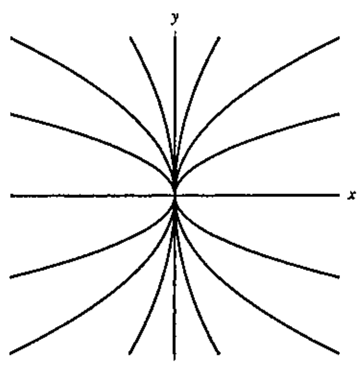
(b) (II) (A) $(x+y^2)=0$

(c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$. (Hint: Process of elimination!)

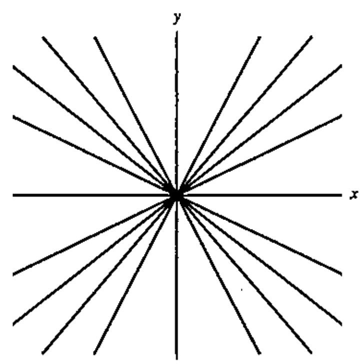
(c) (I) (C)



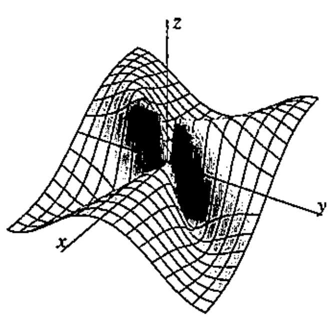
(I)



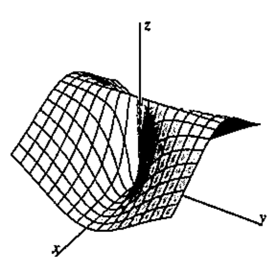
(II)



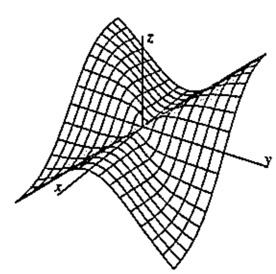
(III)



(A)



(B)



(C)

Definition. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b), but not equal to (a,b). We write this as $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

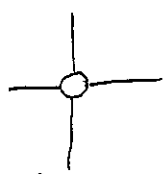
Strategy.

- To show that a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does *not* exist, we usually try to find two different paths approaching (a,b) on which $f(x,y)$ has different limits.
- Showing that a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ *does* exist is generally harder. If the point (a,b) is $(0,0)$, one strategy is to rewrite the limit in polar coordinates. Then, no matter how (x,y) approaches $(0,0)$, r tends to 0, so if the limit $\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta)$ exists, then the original limit $\lim_{(x,y)\to(0,0)} f(x,y)$ also exists.

5. Using the contour maps from #4, first guess whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists for each of the following functions. Then show that your guess is correct using the strategy described above.

(a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$.

Not continuous



$f(x,0) = 0$

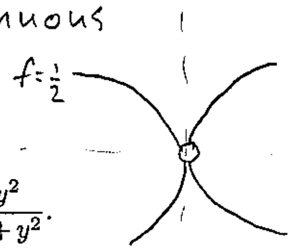
$x \to 0 \Rightarrow f \to 0$

$f(0,y) = 1 \Rightarrow y \to 0 \Rightarrow f \to 1$

So the limit cannot exist

(b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$.

Not continuous



$\frac{1}{2} = f(-x^2, x) \longrightarrow \frac{1}{2}$ as $(-x^2, x) \to 0$

$f = -\frac{1}{2} = f(x^2, x) \longrightarrow -\frac{1}{2}$ as $(x^2, x) \longrightarrow 0$.

(c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$.

This is continuous at 0! Let's show the limit exists.

We can write in polar coords: $f = -\dfrac{r\cos\theta\,(r\sin\theta)^2}{r^2} = -r\sin\theta\cos\theta$

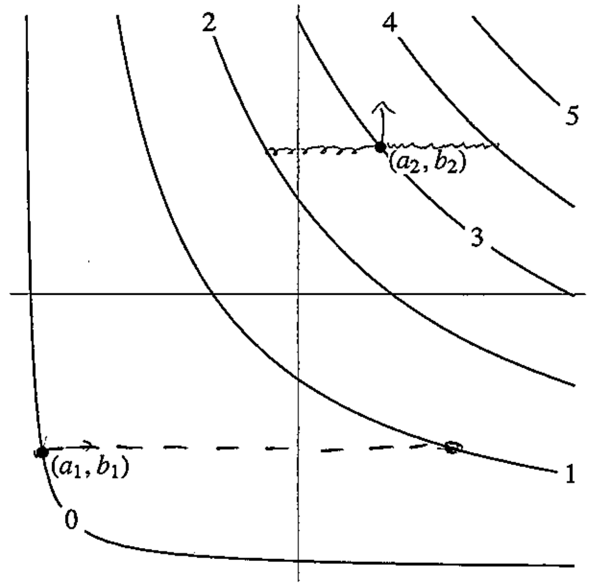
So $|f| \le r \longrightarrow 0$ as $r \to 0$

Therefore, as (r,θ) approaches the origin $(0,0)$ in any direction, f approaches 0

To compute, just imagine all variables are constant except the one you are taking the derivative w.r.t.

Partial Derivatives

Here is the level set diagram of a function $f(x, y)$. The value of f on each level set is labeled.



Eg

$f(x,y) = 3x^2 + 4xy + \cos(y)$

Then

$f_x = \frac{\partial f}{\partial x}(x,y) = 6x + 4y$

$f_y = \frac{\partial f}{\partial y}(x,y) = 4x - \sin y$

$f_{xy} = \frac{\partial f_x}{\partial y} = 4$

$f_{yx} = \frac{\partial f_y}{\partial x} = 4$

When f_{xy} and f_{yx} are defined and continuous, $f_{xy} = f_{yx}$.

Based on the level set diagram, decide whether each of the statements should be true or false. (For which can you be totally sure, and for which would you need more information to be totally sure?)

1. $f_x(a_1, b_1) \geq 0$.

   True

2. $f_y(a_2, b_2) \geq 0$.

   True

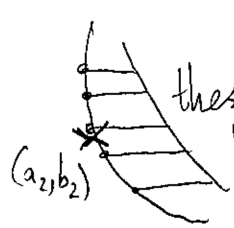
3. $f_x(a_1, b_1) \geq f_x(a_2, b_2)$.

   Can't be certain, but probably False, since ---- is much longer than ~~~ (this means it takes a much larger change in x to increase f by 1 at (a_1, b_1) than at (a_2, b_2).

4. $f_{xx}(a_2, b_2) \geq 0$.

   Compare ~~ ~~~. Honestly, it's hard to tell from this image. Let's say ~~~ is shorter, that means f is "accelerating" when x increases, so $f_{xx}(a_2, b_2) \geq 0$.

5. $f_{xy}(a_2, b_2) \geq 0$.



these horizontal lines lines become shorter as the y-coord increases.

This suggests that f_x is increasing as y increases. In other words $f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} > 0$.