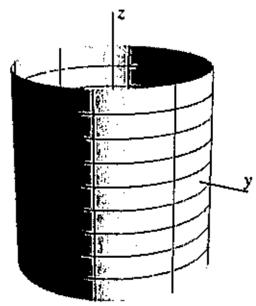
Parametric Surfaces

We can go from cylindrical coordinates (r, θ, z) or spherical coordinates (ρ, θ, ϕ) to Cartesian coordinates (x, y, z) using

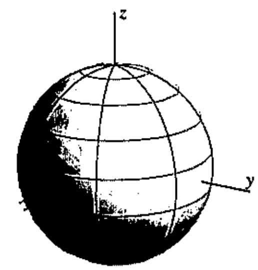
cylindrical	spherical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\phi$

Consider the cylinder in cylindrical coordinates (r, θ, z) with fixed r and the sphere in spherical coordinates (ρ, θ, ϕ) with fixed ρ. We obtain grids by varying one of the remaining coordinates while fixing the other.

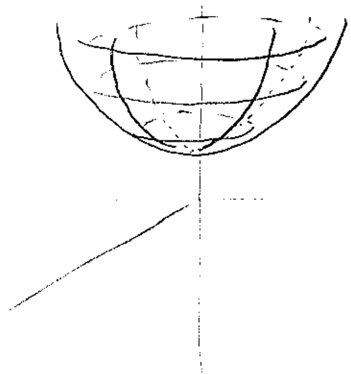
cylindrical $r = 5$



spherical $\rho = 5$



1. (a) Parameterize the elliptic paraboloid $z = x^2 + y^2 + 1$. Sketch the grid curves defined by your parameterization.



$\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 + 1 \rangle$

(b) If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?

$0 \leq \theta \leq 2\pi \qquad r < 3$

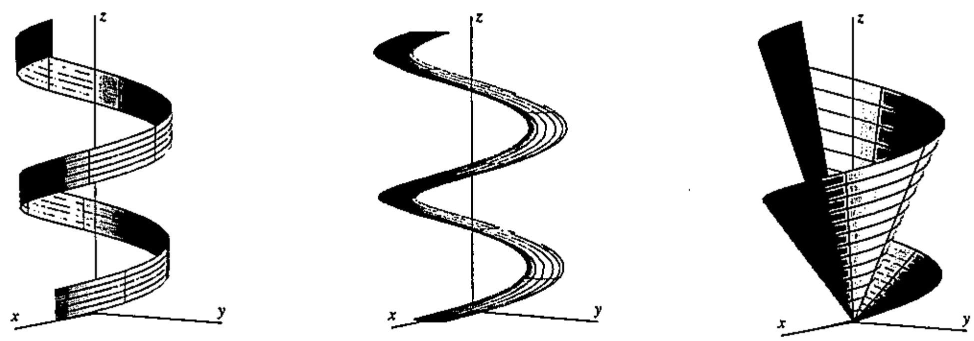
2. Parameterize the plane that contains the 3 points $P(1,0,1)$, $Q(2,-2,2)$, and $R(3,2,4)$.

$\vec{v} = Q - P = (1, -2, 1)$

$\vec{w} = R - P = (2, 2, 3)$

$\vec{r}(s,t) = P + t\vec{v} + s\vec{w} = \langle 1 + t + 2s, -2t + 2s, 1 + t + 3s \rangle$

6. Here are three surfaces.



(I) (II) (III)

Match each function with the surface it parameterizes. Which curves are where u is constant and which curves are where v is constant?

(a) $\vec{r}(u,v) = \left\langle \frac{\cos u}{4} + \cos v, \frac{\sin u}{4} + \sin v, v \right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

II

u horizontal

v the other fellas

(II is the only graph w/ horizontal level curves)

(b) $\vec{r}(u,v) = \left\langle \cos u, \sin u, u + \frac{v}{4} \right\rangle$, $0 \le u \le 4\pi$, $0 \le v \le 2\pi$.

I

u vertical

v the other fellas

(I is the only surface w/ vertical level curves)

(c) $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

III

v the sloped lines

u the others

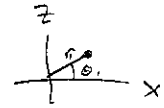
(III has sloped lines)

Strategy: ① fix either u or v and get a parametrization of a curve $\vec{r}_{u=u_0}(v)$ or $\vec{r}_{v=v_0}(u)$

② Hope one of the parametrizations is a line (or something easy to visualize/graph)

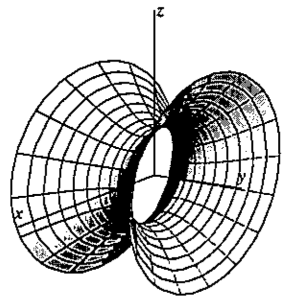
③ Find lines in the image.

Good luck everyone



$x^2+z^2=r_1^2$

3. Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.



$$\vec{r}(r_1, \theta_1) = \left\langle r_1 \cos\theta_1,\ \frac{1}{2}\sqrt{r_1^2-1},\ r_1 \sin\theta_1 \right\rangle$$

$r_1 \geq 1$

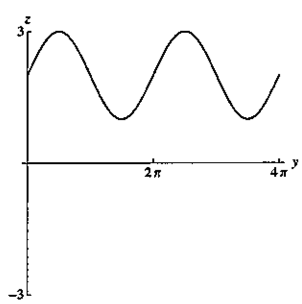
$0 \leq \theta_1 \leq 2\pi$

4. Parameterize the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

~~circle~~ param sphere, then stretch $\qquad \rho = 6$

$$\vec{r}(\theta, \phi) = \left\langle \frac{1}{3}\, 6\cos\theta \sin\phi,\ \frac{1}{2}\, 6 \sin\theta \sin\phi,\ 6\cos\phi \right\rangle$$

5. Consider the curve $z = 2+\sin y$, $0 \leq y \leq 4\pi$ in the yz-plane. Let S be the surface obtained by rotating this curve about the y-axis. Find a parameterization of S.

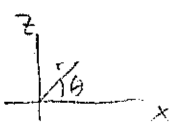


r $\quad \theta \quad$ y $\quad$ coords

$$\vec{r}(y, \theta) = \langle 2+\sin y,\ \theta,\ y \rangle$$

$$= \langle (2+\sin y)\cos\theta,\ y,\ (2+\sin y)\sin\theta \rangle$$

x $\quad$ y $\quad$ z $\quad$ coords



$x = r\cos\theta$
$z = r\sin\theta$