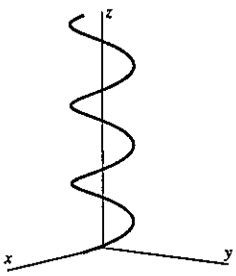
Arc Length and Curvature

1. Last time, we saw that $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ parameterized the pictured curve.



$r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$s(t) = \int_0^t |\sqrt{2}| \, d\tilde{t} = \sqrt{2}\, t$

$\vec{r} = \langle \cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$

(a) Find the arc length of the curve between $(1, 0, 0)$ and $(1, 0, 2\pi)$.

$L = \int_0^{2\pi} |\sqrt{2}| \, dt = 2\sqrt{2}\pi.$

Find param by arc length. $\vec{r}(s) = \langle \cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$

(b) Find the unit tangent vector at the point $(1, 0, 2\pi)$.

$\vec{T}_{(2\pi)} = \frac{\langle 0, 1, 1 \rangle}{\sqrt{2}} = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

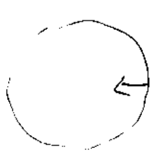
(c) Find the curvature at the point $(1, 0, 2\pi)$.

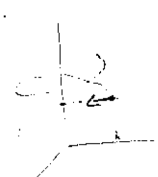
$\kappa(t = 2\pi) = \frac{|\vec{T}_t|}{|\vec{r}_t|} = \frac{|\langle -\sin t, \cos t, 1 \rangle'|}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$

(d) Find the unit normal vector at the point $(1, 0, 2\pi)$.

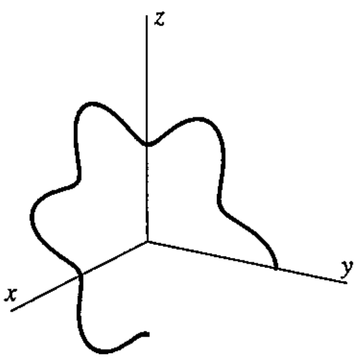
$\frac{\vec{T}_t}{|T_t|} = \langle -\sin t, \cos t, 1 \rangle' = \langle -\cos t, -\sin t, 0 \rangle$

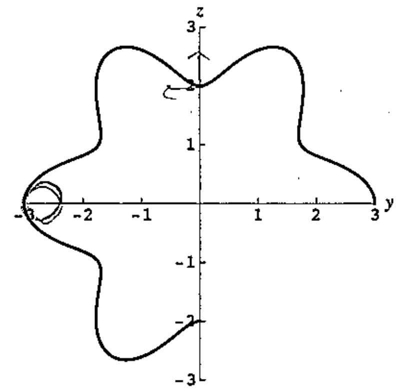
$= \langle -1, 0, 0 \rangle$





2. Suppose that $\vec{r}(t)$, $0 \le t \le 3$, parameterizes the following curve in space, with $\vec{r}(0) = \langle 0, 3, 0 \rangle$ and 3 $\vec{r}(3) = \langle 0, 0, -2 \rangle$. The curve lies entirely in the plane $x = 0$, and the right picture shows just that plane. We are told that the arc length of the curve is approximately 15.3.





Find each of the following, or explain why there is not enough information to do so.

(a) A sketch of the arc length function $s(t)$.

Not enough info

$$\text{Avg} = \frac{15.3}{3} = 5.1$$

(b) The unit tangent vector $\vec{T}$ at the point $(0, 0, 2)$.

$\langle 0, -1, 0 \rangle$

(c) The unit tangent vector $\vec{T}(2)$.

Don't know the param.

(d) The osculating plane at $(0, 0, 2)$.

$\boxed{x = 0}$

(e) The unit normal vector $\vec{N}$ at the point $(0, 0, 2)$. $\langle 0, 0, 1 \rangle$

(f) The unit normal vector $\vec{N}(2)$.

Don't know param.

(g) The binormal vector $\vec{B}$ at the point $(0, 0, 2)$.

$$T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \langle -2, 0, 0 \rangle$$

(h) The normal plane at $(0, 0, 2)$.

Span $\langle N, B \rangle$ = $\boxed{y = 0}$

(i) Which of the following is the best estimate for the curvature of the curve at $(0, -3, 0)$?

$\frac{1}{10}$ $\qquad$ $\frac{1}{2}$ $\qquad$ 2 $\qquad$ 10