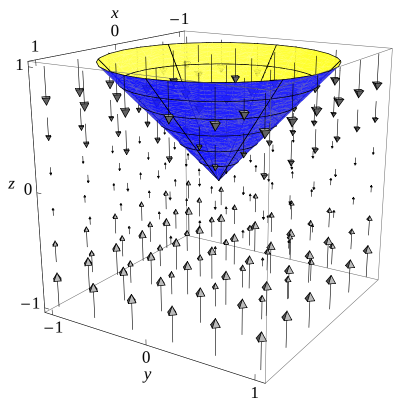
Flux Integrals

The pictures for problems #1 - #4 are on the last page.

1. Let's orient each of the three pictured surfaces so that the light side is considered to be the "positive" side. Decide whether each of the following flux integrals is positive, negative, or zero. ($\vec{F}$ and $\vec{G}$ are the pictured vector fields.)

   (a) $\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$.

   Solution. We want to visualize the surface together with the vector field. Here's a picture of exactly that:

   As we can see, vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_1$ all go from the yellow side to the blue side. (We only care about the vectors that actually go through the surface; so, for instance, we can completely ignore the vectors in the bottom half of the picture since they don't go through the surface.) Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

   (b) $\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S}$.

   Solution. Vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_2$ go from the blue side to the yellow side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "negative" side to the "positive" side, so the flux is $\boxed{\text{positive}}$.

   (c) $\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$.

   Solution. Vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_3$ go from the blue side to the yellow side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "negative" side to the "positive" side, so the flux is $\boxed{\text{positive}}$.

(d) $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$.

Solution. Vectors in the vector field $\vec{G}$ that go through the surface $\mathcal{S}_1$ go from the yellow side to the blue side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

(e) $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$.

Solution. None of the vectors in the vector field $\vec{G}$ actually go *through* the surface $\mathcal{S}_2$, so the flux is $\boxed{\text{zero}}$.

(f) $\iint_{\mathcal{S}_3} \vec{G} \cdot d\vec{S}$.

Solution. Vectors in the vector field $\vec{G}$ that go through the surface $\mathcal{S}_3$ go from the yellow side to the blue side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

2. In each part, you are given an orientation of one of the pictured surfaces. Decide whether this orientation means that the light side or dark side of the surface is the "positive" side, or if the description just doesn't make sense.

(a) $\mathcal{S}_1$, *oriented with normals pointing upward.*

Solution. In order for the normals to be pointing upward, they must be sticking out of the yellow (light) side of the surface. (If they stuck out of the blue side, they would point downward.) So, the $\boxed{\text{yellow (light) side is "positive"}}$.

(b) $\mathcal{S}_2$, *oriented with normals pointing upward.*

Solution. This $\boxed{\text{doesn't make sense}}$. If the normals at the top of the cylinder point up, then the normals at the bottom must point down. There's no way for *all* of the normal vectors to point upward.

(c) $\mathcal{S}_2$, *oriented with normals pointing toward the y-axis.*

Solution. This means that the $\boxed{\text{yellow (light) side is "positive"}}$.

(d) $\mathcal{S}_3$, *oriented with normals pointing outward.*

Solution. This means that the $\boxed{\text{blue (dark) side is "positive"}}$.

(e) $\mathcal{S}_3$, *oriented with normals pointing toward the origin.*

Solution. This means that the $\boxed{\text{yellow (light) side is "positive"}}$.

3. In each part, you are given a parameterization of one of the three pictured surfaces. Decide whether the orientation induced by the parameterization has the light side or dark side of the surface as the

"positive" side.

(a) *For* $\mathcal{S}_1$, $\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2}\rangle$ *with* $u^2+v^2 < 1$.

Solution. To figure out what the normals look like, we'll compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}\vec{r}_u &= \left\langle 1, 0, \frac{u}{\sqrt{u^2+v^2}}\right\rangle \\ \vec{r}_v &= \left\langle 0, 1, \frac{v}{\sqrt{u^2+v^2}}\right\rangle \\ \vec{r}_u \times \vec{r}_v &= \left\langle -\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1\right\rangle\end{aligned}$$

Once we've computed $\vec{r}_u \times \vec{r}_v$, it's not always easy to tell what the resulting orientation looks like. In this particular case, notice that the last component of $\vec{r}_u \times \vec{r}_v$ is just 1, which is of course always positive. This means that all of the normal vectors $\vec{r}_u \times \vec{r}_v$ point upwards. Looking at the picture of the surface $\mathcal{S}_1$, this means that the yellow (light) side is "positive". (Alternatively, we could look back at #2(a), where we looked at this surface with upward normals.)

(b) *For* $\mathcal{S}_1$, $\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$ *with* $0 \le u < 1$ *and* $0 \le v < 2\pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}\vec{r}_u &= \langle \cos v, \sin v, 1\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 0\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -u\cos v, -u\sin v, u\rangle\end{aligned}$$

Notice that the last component of $\vec{r}_u \times \vec{r}_v$ is just u, and our parameterization has $0 \le u < 1$. So, the normal vectors $\vec{r}_u \times \vec{r}_v$ point upwards. Looking at the picture of the surface $\mathcal{S}_1$, this means that the yellow (light) side is "positive".

(c) *For* $\mathcal{S}_2$, $\vec{r}(u,v) = \langle \cos v, u, \sin v\rangle$ *with* $-1 < u < 1$ *and* $0 \le v < 2\pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}\vec{r}_u &= \langle 0, 1, 0\rangle \\ \vec{r}_v &= \langle -\sin v, 0, \cos v\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle \cos v, 0, \sin v\rangle\end{aligned}$$

Here, it's not totally obvious what's going on with the normal vectors. At the point $\vec{r}(u,v) = (\cos v, u, \sin v)$, the normal vector points in the direction $\langle \cos v, 0, \sin v\rangle$. One way to figure out what's going on is just to find the normal vector at a particular point that's easy to visualize. For instance, let's try to find the normal vector at the point $(0,0,1)$. To do this, we need to first find the values of u and v corresponding to this point. That is, we have to solve $\vec{r}(u,v) = \langle 0,0,1\rangle$ for u and v; in this case, $\vec{r}(u,v) = \langle 0,0,1\rangle$ is the same as saying $\cos v = 0$, $u = 0$, and $\sin v = 1$. Therefore, $u = 0$ and $v = \frac{\pi}{2}$. When $u = 0$ and $v = \frac{\pi}{2}$, the normal vector $\vec{r}_u \times \vec{r}_v$ is $\langle 0,0,1\rangle$. This tells us that the normal vector at the point $(0,0,1)$ on top of the cylinder is just $\langle 0,0,1\rangle$, which points upward. Looking at the picture of $\mathcal{S}_2$, this means that the blue (dark) side is "positive".

(d) *For* $\mathcal{S}_3$, $\vec{r}(u,v) = \langle \sin v\cos u, \sin v\sin u, \cos v\rangle$ *with* $0 \le u < 2\pi$ *and* $0 \le v \le \pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle -\sin v \sin u, \sin v \cos u, 0\rangle \\ \vec{r}_v &= \langle \cos v \cos u, \cos v \sin u, -\sin v\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle \end{aligned}$$

Since this looks a bit complicated, let's try to just figure out the normal vector at a point we can understand easily, like the point $(1, 0, 0)$.[1] To figure out the u and v values corresponding to this point, we solve $\vec{r}(u, v) = \langle 1, 0, 0\rangle$ for u and v. This gives three equations:

$$\begin{aligned} \sin v \cos u &= 1 \\ \sin v \sin u &= 0 \\ \cos v &= 0 \end{aligned}$$

By the third equation (and the fact that $0 \leq v \leq \pi$), we know that $v = \frac{\pi}{2}$. Then, $\sin v = 1$, so the first two equations say that $\cos u = 1$ and $\sin u = 0$, which means $u = 0$. So, the point $(1, 0, 0)$ corresponds to $u = 0$, $v = \frac{\pi}{2}$. With these two values, $\vec{r}_u \times \vec{r}_v = \langle -1, 0, 0\rangle$. That is, the normal vector at $(1, 0, 0)$ points toward the origin. From our picture, this means that the yellow (light) side is "positive".

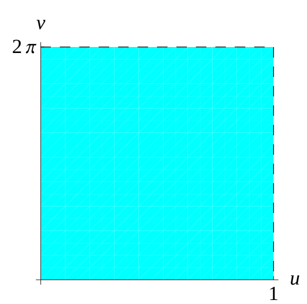
4. Compute the following flux integrals (remember that parameterizations of the surfaces are given in #3). Do the signs of your answers agree with your answers to #1?

(a) $\displaystyle\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$, *where* $\mathcal{S}_1$ *is oriented with normals pointing upward. (*$\vec{F}(x, y, z) = \langle 0, 0, -z\rangle$*, as before.)*

Solution. I like to organize my thinking into a few steps.

- **Step 1 - Parameterize the surface, and figure out the region in the uv-plane describing the possible parameter values.**

  We were given this information in #3; actually, #3(a) and #3(b) gave us two different parameterizations. Let's use the parameterization from #3(b), $\vec{r}(u, v) = \langle u \cos v, u \sin v, u\rangle$ with $0 \leq u < 1$, $0 \leq v < 2\pi$.[2] The region $\mathcal{R}$ in the uv-plane described by $0 \leq u < 1$, $0 \leq v < 2\pi$ is a rectangle:



[1] If you try the point $(0, 0, 1)$ in this example, you'll find that the normal vector is $\langle 0, 0, 0\rangle$. This happens occasionally, and it really doesn't give you any useful information. Just pick another point and try again.

[2] Why did I choose this one instead of the parameterization from #3(a)? Well, in the parameterization from #3(a), the region in the uv-plane describing the possible parameters is a disk; in the parameterization from #3(b), the region in the uv-plane describing the possible parameters is a rectangle, which seems like an easier region of integration to deal with.

- **Step 2 - Decide whether the orientation given by the parameterization matches the desired orientation.**

  In #3(b), we had already figured out that this parameterization had the yellow side as the "positive" side.

  The problem asks us to orient $\mathcal{S}_1$ with its normals pointing upward, which also means that the yellow side should be the "positive" side.

  So, the orientation described by the parameterization does match the desired orientation.

- **Step 3 - Compute!**

  Since the orientation described by our parameterization matches the orientation we want, we know the flux integral is

  $$\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

  To find $\vec{F}(\vec{r}(u,v))$, we just plug our parameterization into $\vec{F}(x,y,z) = \langle 0,0,-z\rangle$, which gives $\vec{F}(\vec{r}(u,v)) = \langle 0,0,-u\rangle$. In #3(b), we found $\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u\rangle$. So, we have

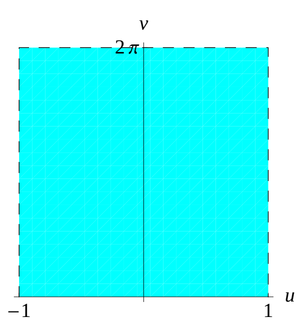
  $$\begin{aligned} \iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{R}} \langle 0,0,-u\rangle \cdot \langle -u\cos v, -u\sin v, u\rangle\ dA \\ &= \iint_{\mathcal{R}} -u^2\ dA \\ &= \int_0^{2\pi} \int_0^1 -u^2\ du\ dv \\ &= \boxed{-\frac{2\pi}{3}} \end{aligned}$$

Note that the sign matches our guess from #1(a); we had said that, if the yellow side was considered the "positive" side (which it is, according to Step 1), then the flux should be negative.

(b) $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$, *where* $\mathcal{S}_2$ *is oriented with normals pointing toward the* y*-axis. (*$\vec{G}(x,y,z) = \langle 0,y,0\rangle$*, as before.)*

Solution. Let's follow the same three steps as in (a):

1. From #3(c), we have the parameterization $\vec{r}(u,v) = \langle \cos v, u, \sin v\rangle$ with $-1 < u < 1$ and $0 \le v < 2\pi$. The region $\mathcal{R}$ in the uv-plane described by these inequalities is a rectangle:



2. In #3(c), we decided that this parameterization had the blue side as the "positive" side, which does *not* match the described orientation (the described orientation has the yellow side as the "positive" side).

3. Since the orientation described by our parameterization does not match the orientation we want, we know the flux integral is

$$\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S} = -\iint_{\mathcal{R}} \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

We already found in #3(c) that $\vec{r}_u \times \vec{r}_v = \langle \cos v, 0, \sin v \rangle$, so

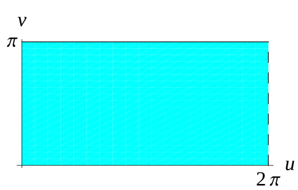
$$\begin{aligned} \iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S} &= -\iint_{\mathcal{R}} \langle 0, u, 0 \rangle \cdot \langle \cos v, 0, \sin v \rangle\ dA \\ &= -\iint_{\mathcal{R}} 0\ dA \\ &= \boxed{0} \end{aligned}$$

Note that this matches our guess from #1(e), where we guessed that the flux would be zero.

(c) $\displaystyle\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$, *where $\mathcal{S}_3$ is oriented with normals pointing outward. ($\vec{F}(x,y,z) = \langle 0, 0, -z \rangle$, as before.)*

Solution. Let's follow the same three steps as in (a):

1. From #3(d), we have the parameterization $\vec{r}(u,v) = \langle \sin v \cos u, \sin v \sin u, \cos v \rangle$ with $0 \leq u < 2\pi$ and $0 \leq v \leq \pi$. The region $\mathcal{R}$ in the uv-plane described by these inequalities is a rectangle:



2. In #3(d), we decided that this parameterization had the yellow side as the "positive" side, which does *not* match the described orientation (the described orientation has the blue side as the "positive" side).

3. Since the orientation described by our parameterization does not match the orientation we want, we know the flux integral is

$$\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S} = -\iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

We already found in #3(d) that $\vec{r}_u \times \vec{r}_v = \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle$, so

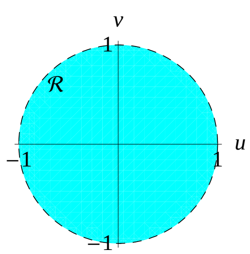
$$\begin{aligned}
\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S} &= -\iint_{\mathcal{R}} \langle 0, 0, -\cos v\rangle \cdot \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle \, dA \\
&= -\iint_{\mathcal{R}} \sin v \cos^2 v \, dA \\
&= -\int_0^{\pi} \int_0^{2\pi} \sin v \cos^2 v \, du \, dv \\
&= -\int_0^{\pi} 2\pi \sin v \cos^2 v \, dv \\
&= \frac{2\pi}{3} \cos^3 v \Big|_{v=0}^{v=\pi} \\
&= \boxed{-\frac{4\pi}{3}}
\end{aligned}$$

Note that the sign matches our answer from #1(c): we had decided that, if the yellow side of the sphere was the "positive" side, then the flux would be positive. This means that, if the blue side of the sphere is "positive" (as it is in this problem), then the flux should be negative.

5. *Let $\mathcal{S}$ be the portion of the surface $3x - 3y + z = 12$ lying inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward. Let $\vec{F}(x, y, z) = \langle -x^2, 0, -3y^2\rangle$. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.*

Solution. Let's follow the same three steps as in #4(a):

1. First, we need to parameterize the surface. If we rewrite the equation $3x - 3y + z = 12$ as $z = 12 - 3x + 3y$, then we see that we can parameterize it by $\vec{r}(u, v) = \langle u, v, 12 - 3u + 3v\rangle$. We want only the portion with $x^2 + y^2 < 1$. In terms of our parameters, this says $u^2 + v^2 < 1$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is the disk $u^2 + v^2 < 1$:



2. Next, we need to see what orientation this parameterization describes, as we did in #3. First, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}
\vec{r}_u &= \langle 1, 0, -3\rangle \\
\vec{r}_v &= \langle 0, 1, 3\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle 3, -3, 1\rangle
\end{aligned}$$

This always points upward, so this matches the orientation we want (after all, we are told that the normals should point upward).

3. Now, we just compute. Since the orientation given by $\vec{r}(u, v)$ matches the orientation we want,

we know the flux integral is

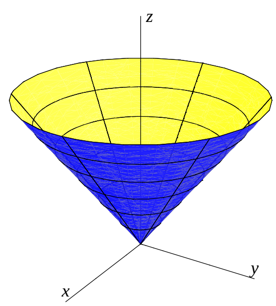
$$\begin{aligned}\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -u^2, 0, -3v^2 \rangle \cdot \langle 3, -3, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} (-3u^2 - 3v^2)\ dA\end{aligned}$$

Since $\mathcal{R}$ is a disk, this integral will be easier to do in polar coordinates. In polar coordinates (thinking of u and v as x and y), the region is $0 \le r < 1$, $0 \le \theta < 2\pi$, so the integral is

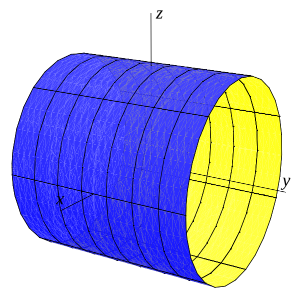
$$\begin{aligned}\int_0^{2\pi} \int_0^1 -3r^2 \cdot r\ dr\ d\theta &= \int_0^{2\pi} \left(-\frac{3}{4} r^4 \Big|_{r=0}^{r=1} \right) d\theta \\ &= \int_0^{2\pi} -\frac{3}{4}\ d\theta \\ &= \boxed{-\frac{3}{2}\pi}\end{aligned}$$

These are the surfaces for problems #1 - #4. Each is colored so that one side of the surface is light and the other side is dark.

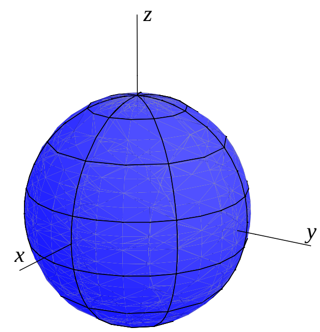
$\mathcal{S}_1$ is the portion of the cone $z = \sqrt{x^2 + y^2}$ under the plane $z = 1$.



$\mathcal{S}_2$ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = -1$ and $y = 1$.

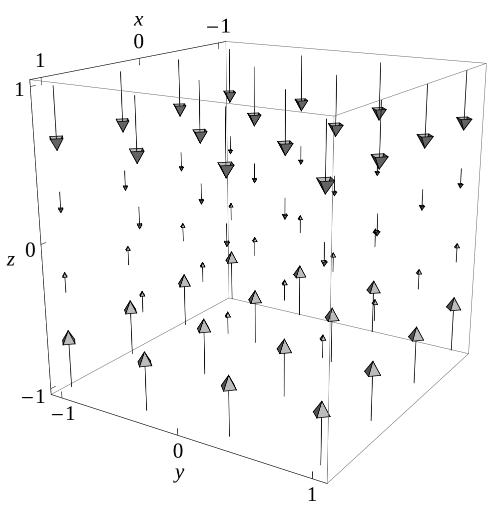


$\mathcal{S}_3$ is the unit sphere $x^2 + y^2 + z^2 = 1$.



These are the vector fields $\vec{F}$ and $\vec{G}$ for problems #1 - #4. (Note that the origin is located in the middle of each box.)

$\vec{F}(x, y, z) = \langle 0, 0, -z \rangle$



$\vec{G}(x, y, z) = \langle 0, y, 0 \rangle$