Green's Theorem

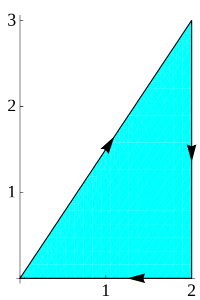
1. *Let C be the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$, oriented counterclockwise, and let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle e^y + x, x^2 - y\rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$.*

   Solution. Let's write $P(x,y) = e^y + x$ and $Q(x,y) = x^2 - y$, so that $\vec{F} = \langle P, Q\rangle$. Let $\mathcal{R}$ be the region $0 \le x \le 1$, $0 \le y \le 1$. The boundary of $\mathcal{R}$, oriented "correctly" (so that a penguin walking along it keeps $\mathcal{R}$ on his left), is the given curve C. So, Green's Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\, dA = \iint_{\mathcal{R}} (2x - e^y)\, dA$. We compute this by converting it to an iterated integral:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_{\mathcal{R}} (2x - e^y)\, dA \\
&= \int_0^1 \int_0^1 (2x - e^y)\, dx\, dy \\
&= \int_0^1 \left(x^2 - xe^y \Big|_{x=0}^{x=1} \right) dy \\
&= \int_0^1 (1 - e^y)\, dy \\
&= y - e^y \Big|_{y=0}^{y=1} \\
&= \boxed{2 - e}
\end{aligned}$$

2. *Let C be the oriented curve consisting of line segments from $(0,0)$ to $(2,3)$ to $(2,0)$ back to $(0,0)$, and let $\vec{F}(x,y) = \langle y^2, x^2\rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$.*

   Solution. Here is a picture of the curve C, along with the interior of the triangle, which we'll call $\mathcal{R}$:

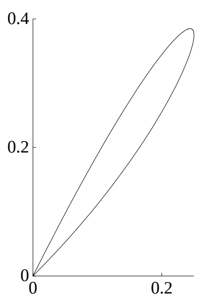


   The boundary of $\mathcal{R}$, oriented "correctly" (so that a penguin walking along it keeps $\mathcal{R}$ on his left side), is $-C$ (that is, it's C with the opposite orientation). So, Green's Theorem says that $\int_{-C} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\, dA$, where $F = \langle P, Q\rangle$. We are looking for $\int_C \vec{F} \cdot d\vec{r}$, which we know is the negative of

$\int_{-C} \vec{F} \cdot d\vec{r}$. Therefore,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= -\iint_{\mathcal{R}} (Q_x - P_y)\ dA \\
&= -\iint_{\mathcal{R}} (2x - 2y)\ dA \\
&= -\int_0^2 \int_0^{3x/2} (2x - 2y)\ dy\ dx \\
&= -\int_0^2 \left(2xy - y^2 \Big|_{y=0}^{y=3x/2} \right)\ dx \\
&= -\int_0^2 \frac{3}{4}x^2\ dx \\
&= -\left(\frac{1}{4}x^3 \Big|_{x=0}^{x=2} \right) \\
&= \boxed{-2}
\end{aligned}$$

3. *Find the area of the region enclosed by the parameterized curve* $\vec{r}(t) = \langle t - t^2, t - t^3 \rangle$, $0 \le t \le 1$.

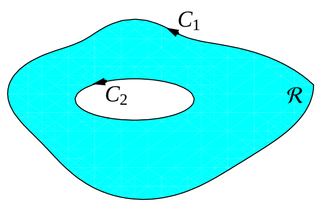


Solution. Let $\mathcal{R}$ be the region in question. We know from #2(a) on the worksheet "Double Integrals" that the area of $\mathcal{R}$ is $\iint_{\mathcal{R}} 1\ dA$. Normally, we would evaluate this by converting it to an iterated integral; in this case, that's quite challenging, so we'll instead use Green's Theorem to evaluate this integral. If we can come up with a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ such that $Q_x - P_y = 1$, then Green's Theorem will say that $\iint_{\mathcal{R}} 1\ dA = \int_C \vec{F} \cdot d\vec{r}$, where C is the boundary of the region, traveled counterclockwise (so that a penguin walking along C keeps $\mathcal{R}$ on his left). One such vector field is $\vec{F}(x, y) = \langle 0, x \rangle$.

We are given a parameterization $\vec{r}(t)$ of the curve, and this parameterization does in fact travel the curve counterclockwise.[1] So,

$$\begin{aligned}
\iint_{\mathcal{R}} 1\ dA &= \int_C \vec{F}\cdot d\vec{r} \\
&= \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\ dt \\
&= \int_0^1 \langle 0, t-t^2\rangle \cdot \langle 1-2t, 1-3t^2\rangle\ dt \\
&= \int_0^1 (t-t^2)(1-3t^2)\ dt \\
&= \int_0^1 (t-t^2-3t^3+3t^4)\ dt \\
&= \frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{3}{4}t^4 + \frac{3}{5}t^5\Big|_{t=0}^{t=1} \\
&= \boxed{\frac{1}{60}}
\end{aligned}$$

4. *Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be any vector field defined on the region $\mathcal{R}$ (in $\mathbb{R}^2$) shown in the picture, and let C_1 and C_2 be the oriented curves shown in the picture. What does Green's Theorem say about $\int_{C_1} \vec{F}\cdot d\vec{r}$, $\int_{C_2} \vec{F}\cdot d\vec{r}$, and $\iint_{\mathcal{R}} (Q_x - P_y)\ dA$?*



Solution. The boundary of $\mathcal{R}$ consists of two curves, C_1 and C_2. A penguin walking along C_1 in the indicated direction would indeed keep $\mathcal{R}$ on his left, but a penguin walking along C_2 in the indicated direction would have $\mathcal{R}$ on his right. So, the boundary of $\mathcal{R}$ is really C_1 together with $-C_2$, which means $\boxed{\iint_{\mathcal{R}} (Q_x - P_y)\ dA = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}}$.

5. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$. You can check that $P_y = Q_x$.

   (a) *What is wrong with the following reasoning? "$P_y = Q_x$, so $\vec{F}$ is conservative."*

   Solution. $\vec{F}$ is not defined at the origin, so its domain is $\mathbb{R}^2$ except the point $(0,0)$. This domain is not simply connected, so we cannot conclude anything from the fact that $P_y = Q_x$.

   (b) *Let C be any simple closed curve in $\mathbb{R}^2$ that does not enclose the origin, oriented counterclockwise.*

[1] This is not completely obvious, but there's an easy way to tell at the end whether the parameterization went the right way — we are looking for an area, so our final answer must be positive.

(A simple curve is a curve that does not cross itself.) Use Green's Theorem to explain why $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution. Since C does not go around the origin, $\vec{F}$ is defined on the interior $\mathcal{R}$ of C. (The only point where $\vec{F}$ is not defined is the origin, but that's not in $\mathcal{R}$.) Therefore, we can use Green's Theorem, which says $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\ dA$. Since $Q_x - P_y = 0$, this says that $\int_C \vec{F} \cdot d\vec{r} = 0$.

(c) *Let a be a positive constant, and let C be the circle $x^2 + y^2 = a^2$, oriented counterclockwise. Parameterize C (check your parameterization by plugging it into the equation $x^2 + y^2 = a^2$), and use the definition of the line integral to show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Why doesn't the reasoning from (b) work in this case?)*

Solution. One possible parameterization of C is $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$, $0 \le t \le 2\pi$. Then,

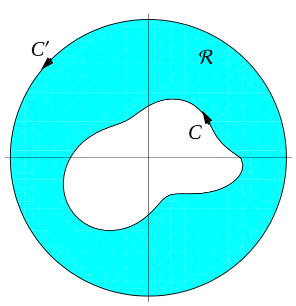
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt \\
&= \int_0^{2\pi} \left\langle \frac{a\cos t}{\sqrt{(a\cos t)^2 + (a\sin t)^2}}, \frac{a\sin t}{\sqrt{(a\cos t)^2 + (a\sin t)^2}} \right\rangle \cdot \langle -a\sin t, a\cos t\rangle\ dt \\
&= \int_0^{2\pi} 0\ dt \\
&= 0,
\end{aligned}$$

as we wanted.

We cannot use the reasoning from (b) since $\vec{F}$ is not defined in the whole interior of C (in particular, it's not defined at the origin, which is inside C).

(d) *Let C be any simple closed curve in $\mathbb{R}^2$ that does enclose the origin, oriented counterclockwise. Explain why $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: Use (c) and #4.)*

Solution. No matter what C looks like, we can draw a giant circle $x^2 + y^2 = a^2$ around the origin that encloses all of C. Let's orient this giant circle counterclockwise and call it C', and let's have $\mathcal{R}$ be the region between C and C' $\mathcal{R}$:



Notice that $\vec{F}$ is defined on all of $\mathcal{R}$ (because it is defined everywhere except the origin, and $\mathcal{R}$

doesn't include the origin). So, #4 tells us that

$$\iint_{\mathcal{R}} (Q_x - P_y)\ dA = \int_{C'} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}.$$

We showed in (c) that $\int_{C'} \vec{F} \cdot d\vec{r} = 0$, so this simplifies to

$$\iint_{\mathcal{R}} (Q_x - P_y)\ dA = -\int_C \vec{F} \cdot d\vec{r}.$$

Since $Q_x = P_y$ inside of $\mathcal{R}$, the double integral is really a double integral of 0, so it's equal to 0. Therefore, we conclude that $\int_C \vec{F} \cdot d\vec{r} = 0$ as well.

(e) *Is it valid to conclude from the above reasoning that, if $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector field defined everywhere except the origin and $P_y = Q_x$, then $\vec{F}$ is conservative?*

Solution. No! The calculation in (c) only applied to this particular vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.

There are vector fields that are defined everywhere except the origin and satisfy $P_y = Q_x$ but are still not conservative; the vector field in #4(b) of the worksheet "The Fundamental Theorem for Line Integrals; Gradient Vector Fields" is an example.

6. In this problem, you'll prove Green's Theorem in the case where the region is a rectangle. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field on the rectangle $\mathcal{R} = [a,b] \times [c,d]$ in $\mathbb{R}^2$.

(a) *Show that* $\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx.$

Solution. Let's first break the given double integral into a difference of two double integrals:

$$\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \iint_{\mathcal{R}} Q_x(x,y)\ dA - \iint_{\mathcal{R}} P_y(x,y)\ dA.$$

Now, we'll convert the double integrals on the right side to iterated integrals. This is easy, since the region $\mathcal{R}$ is just a rectangle. However, we're going to do the two iterated integrals in different orders: it makes sense to first integrate Q_x with respect to x (since it's a derivative with respect to x) and to first integrate P_y with respect to y:

$$\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \int_c^d \int_a^b Q_x(x,y)\ dx\ dy - \int_a^b \int_c^d P_y(x,y)\ dy\ dx.$$

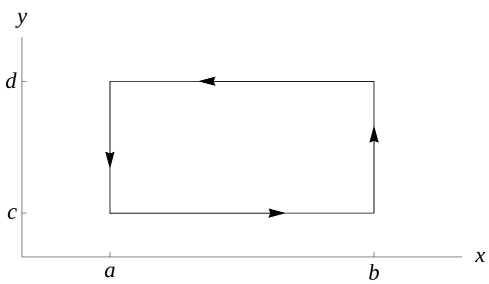
When we integrate Q_x with respect to x, we just get Q; similarly, when we integrate P_y with respect to y, we just get P:

$$\begin{aligned}\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA &= \int_c^d \left(Q(x,y)\Big|_{x=a}^{x=b} \right) dy - \int_a^b \left(P(x,y)\Big|_{y=c}^{y=d} \right) dx \\ &= \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx,\end{aligned}$$

which is exactly what we were asked to show.

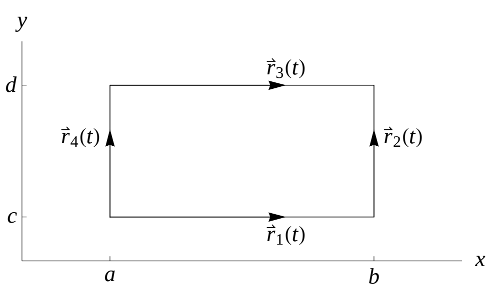
(b) *Let C be the boundary of $\mathcal{R}$, traversed counterclockwise. Show that* $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ *is also equal to* $\displaystyle\int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx.$

Solution. Here is a picture of C:



As we can see, it's composed of 4 pieces, and we'll parameterize each separately. The bottom piece has $y = c$, so only x varies, and we can parameterize it using $\vec{r}_1(t) = \langle t, c\rangle$ with $a \le t \le b$. The right piece has $x = b$, so only y varies, and we can parameterize it using $\vec{r}_2(t) = \langle b, t\rangle$, $c \le t \le d$.

The top piece has $y = d$, so only x varies, and we'd like to parameterize it using $\vec{r}_3(t) = \langle t, d\rangle$. The slight problem with this is that it goes the wrong direction: as t increases, $\langle t, d\rangle$ goes to the right. This is actually not a problem, as long as we account for it later. So, we'll go ahead and use $\vec{r}_3(t) = \langle t, d\rangle$ with $a \le t \le b$. Similarly, for the left piece, we'll use $\vec{r}_4(t) = \langle a, t\rangle$, $c \le t \le d$. Here's a diagram showing the various things we've parameterized:



As we can see from the two diagrams,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1(t)} \vec{F} \cdot d\vec{r} + \int_{\vec{r}_2(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_3(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_4(t)} \vec{F} \cdot d\vec{r}.$$

Plugging the four parameterizations into this, $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is equal to

$$\int_a^b \vec{F}(t,c) \cdot \langle 1, 0\rangle\ dt + \int_c^d \vec{F}(b,t) \cdot \langle 0, 1\rangle\ dt - \int_a^b \vec{F}(t,d) \cdot \langle 1, 0\rangle\ dt - \int_c^d \vec{F}(a,t) \cdot \langle 0, 1\rangle\ dt.$$

Writing $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, we can simplify this to

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P(t,c)\ dt + \int_c^d Q(b,t)\ dt - \int_a^b P(t,d)\ dt - \int_c^d Q(a,t)\ dt.$$

This is exactly what we were supposed to show, which is more obvious if we rename t to be x in the first and third integrals, rename t to be y in the second and fourth integrals, and rearrange the terms:

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_a^b P(x,c)\ dx + \int_c^d Q(b,y)\ dy - \int_a^b P(x,d)\ dx - \int_c^d Q(a,y)\ dy \\
&= \int_c^d Q(b,y)\ dy - \int_c^d Q(a,y)\ dy - \int_a^b P(x,d)\ dx + \int_a^b P(x,c)\ dx \\
&= \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx
\end{aligned}$$