STOKE'S THEOREM: THE FUNDAMENTAL THEOREM OF CALCULUS (IN MULTIPLE VARIABLES)

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1. Vector fields

Definition 1.1. A *vector field* is a section of $T\mathbb{R}^n \to \mathbb{R}^n$. In other words, it is an assignment of a vector to every point in the domain.

Example 1.2. We have already seen vector fields. The gradient of a function ∇f is a vector field.

Definition 1.3. A *flow line* of a vector field $\vec{F}$ is a path $\vec{r}(t)$ whose velocity at every point in the trajectory is $\vec{F}$. That is, $\vec{r}'(t) = \vec{F}$.

Remark 1.4. For every vector field $\vec{F}$ and every point P in the domain, there is a flow line of $\vec{F}$ originating at P.

Definition 1.5. A function f is the *potential* of the vector field $\vec{F}$ if $\nabla f = \vec{F}$.

2. Something cool

We can actually integrate vector fields.

Definition 2.1 (Line integral)**.** The integral of the vector field $\vec{F}$ along a curve C is

$$\int_C \vec{F} = \int_a^b \vec{F} \cdot \frac{\partial \vec{r}}{\partial t} dt \tag{1}$$

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where C is parameterized by $\vec{r}(t)$ for $t \in [a, b]$.

Remark 2.2. This definition and notation suggests that the integral is independent of parametrization. This is true, because if we choose a new parametrization $\vec{r}(s)$, then

$$\int \vec{F} \cdot \frac{\partial \vec{r}}{\partial s} ds = \int \vec{F} \cdot \frac{\partial \vec{r}}{\partial s}\frac{\partial s}{\partial t} dt = \int \vec{F} \cdot \frac{\partial \vec{r}}{\partial t} dt. \tag{2}$$

3. Crazy revelation

Actually, we have **always** and **only** been integrating vector fields. The dt that single-variable calculus teachers try to avoid is actually a vector field!

Example 3.1. For starters, consider the single-variable function x on $\mathbb{R}$. An interval $I = [a, b] \subset \mathbb{R}$ can be thought of as a curve. The gradient is $\nabla x = \langle 1 \rangle$ is a vector field on $\mathbb{R}$.

$$\int_I \nabla x = \int_a^b dx. \tag{3}$$

In other words, dx is really just a gradient ∇x.

Let $f(x)$ be a function of a single variable. Then

$$\int_I \nabla f = \int_a^b f' dx \tag{4}$$

4. Types of vector fields

Since we integrate vector fields over 1-dimensional objects, we refer to them as 1-vector fields (abbreviated as 1-vf).

In general, there is a *type d-vector field* (abbreviated d-vf) which we integrate over d-dimensional objects. Since this class is based in $\mathbb{R}^3$, d-vfs for $d = 0, 1, 2, 3$ are described below.

(1) 0-vf is a function f, associating a scalar value to every point in $\mathbb{R}^3$.
(2) 1-vf is a vector-valued function $\vec{F}$, associating a vector to each point.
(3) 2-vf is a vector valued function $\vec{\varphi}$, associating a vector to each point.
(4) 3-vf is a function Φ, associating a scalar value to every point.

Note 4.1. Notice the symmetry: the outer dimensional vfs are standard functions while the inner dimensional vfs are standard vector fields.

5. Integrals of d-vector fields

Here we describe how to integrate d-vfs over d-dimensional objects in $\mathbb{R}^3$ for $d = 0, 1, 2, 3$.

(1) (Sum) Let $P = p_1, \ldots, p_n$ be a collection of points (0-dim) and f is a 0-vf defined on P. Then

$$\int_P f = \sum_i f(p_i). \tag{5}$$

(2) (Line integral) Let C be a curve and $\vec{F}$ a 1-vf. Then $\int_C \vec{F}$ is described above.

$$\int_C \vec{F} = \int \vec{F} \cdot \vec{r}'(t) dt \tag{6}$$

as above.

(3) (Flux integral) Let S be a surface and $\vec{\varphi}$ a 2-vf. Then

$$\int_S \vec{\varphi} = \int_S \vec{\varphi} \cdot d\vec{A} = \int_S \vec{\varphi} \cdot \vec{n} dA = \int \int \vec{\varphi} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| dudv = \int \int [\vec{\varphi}, \vec{r}_u, \vec{r}_v] dudv \tag{7}$$

where $\vec{r}(u, v)$ is any parametrization of S.

(4) (Volume integral) Let E be a solid and Φ a 3-vf. Then

$$\int_E \Phi = \int_E \Phi dV. \tag{8}$$

Note 5.1 (CAUTION)**.** Signs will be affected by *orientation*.

6. Derivatives of vector fields

There is a way to take the derivative of a d-vf to obtain a $(d+1)$-vf. We already know the 0^{th} derivative.

Example 6.1. The derivative of a 0-vf is the gradient. That is, a 0-vf is a function f and its derivative is the 1-vf ∇f.

Definition 6.2. The derivative of a 1-vf $\vec{F} = \langle P, Q, R \rangle$ is

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \tag{9}$$

and is called the *curl*.

Definition 6.3. The derivative of a 2-vf $\vec{\varphi} = \langle \alpha, \beta, \gamma \rangle$ is

$$\text{div}(\vec{\varphi}) = \nabla \cdot \vec{\varphi} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \vec{\varphi} = \alpha_x + \beta_y + \gamma_z \tag{10}$$

and is called the *divergence* of $\vec{\varphi}$.

7. Single-variable recap: FTC

Let $f(x)$ be a function of one variable on $I = [a, b] \subset \mathbb{R}$ an interval. The fundamental theorem of calculus says

Theorem 7.1. $\int_a^b f'(x)dx = f(b) - f(a)$

This is what we use to compute almost every integral. Another way to say this is,

Theorem 7.2 (Fundamental theorem of calculus)**.** $\int_I \nabla f = \int_{\partial I} f$.

Integrating the derivative of a 0-vector field on a curve is the same as integrating the 0-vector field on the boundary.

8. FTC in 1, 2, and 3 dimensions

Let C be a curve (1-dim) with boundary ∂C (0-dim) and f a 0-vf. Then

Theorem 8.1 (Fundamental theorem of line integrals)**.** $\int_C \nabla f = \int_{\partial C} f$.

Let S be a surface (2-dim) with boundary ∂S (1-dim) and $\vec{F}$ a 1-vf. Then

Theorem 8.2 (Green's theorem)**.** $\int_S \nabla \times \vec{F} = \int_{\partial S} \vec{F}$.

Let E be a solid (3-dim) with boundary ∂E (2-dim) and $\vec{\varphi}$ a 2-vf. Then

Theorem 8.3 (Divergence theorem)**.** $\int_E \nabla \cdot \vec{\varphi} = \int_{\partial E} \vec{\varphi}$.

9. Stoke's theorem

All of the amazing theorems in the previous two sections are special cases of a very general theorem.

Theorem 9.1 (Stoke's theorem)**.** *Let M be a d-dimensional space with $(d-1)$-dimensional boundary ∂M. Let ω be a $(d-1)$-vf with derivative $\nabla\omega$. Then*

$$\int_M \nabla\omega = \int_{\partial M} \omega. \tag{11}$$

10. Properties of vector fields

The vf derivatives have an intriguing property.

Theorem 10.1. *The derivative of a derivative of a d-vf is* 0.

Example 10.2. We have two important examples of this in $\mathbb{R}^3$ (which you should compute and check yourself). Let f be a 0-vf and $\vec{F}$ a 1-vf. Then $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times \vec{F} = 0$.

Definition 10.3. A vector field $\vec{F}$

1. is a *gradient field* if it is the gradient of a function $\vec{F} = \nabla f$.
2. is *conservative* if $\int_C \vec{F}$ depends only on the endpoints of C.
3. satisfies the *closed loop property* if $\int_C \vec{F} = 0$ for any closed loop C.
4. is *irotational* if $\nabla \times \vec{F} = 0$.

Theorem 10.4. *The first three are equivalent and imply irotational. Irotational implies the first three on simply connected domains.*

Proof. There is a circle of implications.

1. (closed implies conservative) If C_1 and C_2 are paths from point P to point Q, then $C_1 - C_2$ is a closed loop. Therefore,
$$0 = \int_{C_1 - C_2} \vec{F} = \int_{C_1} \vec{F} - \int_{C_2} \vec{F},$$
so $\int_{C_1} \vec{F} = \int_{C_2} \vec{F}$.
2. (conservative implies gradient) Fix a point P and define the function f by $f(Q) := \int_P^Q \vec{F}$. FTC says
$$\int_C \nabla f = \int_{\partial C} f = \int_{\partial C} \int_P^* \vec{F} = \int_C \vec{F}$$
for every curve C, so $\nabla f = \vec{F}$.
3. (gradient implies closed) If $\vec{F} = \nabla f$, then $\int_C \vec{F} = \int_{\partial C} f = 0$ since $\partial C = \varnothing$.
4. (gradient implies irotational) $\nabla \times \nabla f = 0$.
5. (irotational implies closed (in simply connected domains)) Let C be any closed curve, which necessarily bounds a disk D. Green's theorem says
$$\int_C \vec{F} = \int_D \nabla \times \vec{F} = 0.$$

$\square$