A PROOF OF THE BOREL-WEIL-BOTT THEOREM

JACOB LURIE

The aim of this note is to provide a quick proof of the Borel-Weil-Bott theorem, which describes the cohomology of line bundles on flag varieties. Let G denote a reductive algebraic group over the field $\mathbf{C}$ of complex numbers. We let B denote a Borel subgroup of G, and $X = G/B$ be quotient of G by B, acting by right multiplication. The quotient X is a compact complex manifold. The anticanonical bundle of X is ample, so that X has the structure of a projective algebraic variety.

Our actual objects of study are not line bundles on X, but equivariant line bundles on X. The following result is, strictly speaking, unnecessary for what follows. Nevertheless it is an interesting fact in its own right:

Theorem 1. *If G is semisimple and simply connected, then the forgetful functor from the groupoid of equivariant line bundles on X to the groupoid of line bundles on X is an equivalence of categories.*

Proof. The forgetful functor is faithful by construction. To see that it is full, we must show that every isomorphism $\mathcal{L} \to \mathcal{L}'$ of equivariant line bundles is actually G-equivariant. To specify such an isomorphism is to specify a nowhere vanishing section of the equivariant line bundle $\mathcal{L}^{-1} \otimes \mathcal{L}'$. This line bundle is trivial, hence G acts by a character on its space of global sections. Since G is semisimple, this character must be trivial, and any global section is invariant.

It remains to show that the forgetful functor is essentially surjective. In other words, we must show that any line bundle $\mathcal{L}$ on X admits an action of G, compatible with the action of G on X. Let $a : G \times X \to X$ classify the action of G on X, and let

$$\pi_1 : G \times X \to G$$

$$\pi_2 : G \times X \to X$$

be the projections. Let $\mathcal{M} = a^*\mathcal{L} \otimes \pi_2^*\mathcal{L}^{-1}$. Since X is simply connected (as one sees, for example, from the Bruhat decomposition, which exhibits X as a CW-complex with no 1-cells), its Picard variety is trivial. Thus the fibers of a connected family of line bundles on X are all isomorphic. In particular, since the fiber of $\mathcal{M}$ over the identity $e \in G$ is trivial on $\{e\} \times X$, $\mathcal{M}$ is trivial on each fiber of π_1. Since π_1 is proper, it follows that $\mathcal{M}$ is a pullback of a line bundle on G.

Since $\mathrm{H}^2(G, \mathbf{Z})$ is trivial, any line bundle on G is topologically trivial. As G is a Stein space (it is an affine algebraic variety), such a line bundle is also holomorphically trivial. Thus $\mathcal{M}$ is (holomorphically) trivial, so $a^*\mathcal{L} \simeq \pi_2^*\mathcal{L}$ as holomorphic line bundles. Consequently there exists a projective action of G on $\mathcal{L}$ compatible with its action on X. To this projective action there corresponds a central extension $\widetilde{G}$ of G by $\mathbf{C}^\times$. Since G is simply-connected, the structure theory of reductive Lie groups shows that this extension is split (the commutator subgroup of $\widetilde{G}$ maps isomorphically onto G). Thus there is an action of G on $\mathcal{L}$, compatible with its action on X. ■

Equivariant line bundles on X are easily described. They are classified by the one-dimensional (holomorphic) representations of B. Let U denote the unipotent radical of B and T a maximal torus of B, so that $B = TU$. Then one-dimensional representations of B correspond bijectively to the characters of T. Let Λ denote the character group of T, and for each $\lambda \in \Lambda$, let $\mathcal{L}_\lambda$ denote the corresponding equivariant line bundle. Let W denote the Weyl group of G.

Theorem 2. *If $-\lambda$ is not a dominant weight, then $\mathrm{H}^0(X, \mathcal{L}_\lambda) = 0$. If $-\lambda$ is dominant, then $\mathrm{H}^0(X, \mathcal{L}_\lambda)$ is an irreducible representation of G with lowest weight λ (and therefore dual to the representation of G with highest weight $-\lambda$).*

Proof. Since X is compact, $V = \mathrm{H}^0(X, \mathcal{L}_\lambda)$ is a finite-dimensional (holomorphic) representation of G. Let U' be the unipotent radical of a Borel subgroup opposite B containing the torus T, and let $V_0 \subseteq V$ denote the subspace of "lowest weight" vectors: that is, the subset consisting of elements which are invariant under U'.

We can identify V with the space of holomorphic functions on G satisfying the functional equation $v(xb) = \lambda(b)v(x)$ for each $b \in B$. In this representation, G acts by left multiplication: $(gv)(x) = v(gx)$. We may therefore identify the V_0 with the space of holomorphic functions in V which also satisfy $v(ux) = v(x)$ for $u \in U'$. In particular, we have $v(ub) = \lambda(b)v(1)$ for $u \in U'$, $b \in B$. Since $U'B$ is dense in G, we see that V_0 is at most one-dimensional. If nonzero, it is generated by a single function v satisfying $v(1) = 1$. For $t \in T$, we have

$$(tv)(1) = v(t) = \lambda(t)v(1),$$

so that $tv = \lambda(t)v$ and therefore V is an irreducible representation with lowest weight λ. Clearly, this is impossible unless $-\lambda$ is dominant.

All that remains is to prove that V_0 is nontrivial if $-\lambda$ is dominant. In this case, we claim that there is a holomorphic function v on G such that $v(ub) = \lambda(b)$. Then v satisfies the conditions

$$v(gb) = \lambda(b)v(g)$$

$$v(ug) = v(g)$$

for all $g \in U'B$, hence for all $g \in G$ by continuity, so that $v \in V_0$ as desired.

To construct the function v, let $B' = U'T$ be a Borel subgroup opposite B, and consider the Bruhat decomposition of G into double cosets $B'\backslash G/B$. The "big cell" of this decomposition is biholomorphic to $U' \times B$, so the condition $v(ub) = \lambda(b)$ determines v uniquely on $U'B$. We must show that this holomorphic function extends over the other Bruhat cells. Since v is algebraic on $U'B$, it suffices to show that v has no poles along Bruhat cells of codimension 1 in G. Let C be such a cell, corresponding to a simple root α. If v has a pole along C, then since v has no zeros on $U'B$, it approaches infinity near every point of C. Write $C \cup U'B = SB$, where S is a subgroup of G generated by U' and the unipotent subgroup corresponding to a single positive root. Let G' denote the universal cover of a Levy factor of S which has a maximal torus contained in T. Then G' is isomorphic to $\mathrm{SL}_2(\mathbf{C})$. Pulling everything back to G', we reduce to proving the statement in the case $G = G'$.

Now the problem is much more concrete. We may identify G with the space of all two-by-two matrices with determinant 1, B to consist of all upper-triangular matrices, and U' all unipotent lower triangular matrices. If we then identify T with all diagonal matrices, then the character λ is given by

$$\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mapsto t^k$$

for some integer k. The function v is then given, on the big cell, by the formula

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto a^k$$

Clearly, this function extends holomorphically to all of G if and only if $k \geq 0$; that is, if and only if $-\lambda$ is dominant. ■

We now need to closely examine the cohomology of line bundles in the simplest special case, $G = \mathrm{SL}_2(\mathbf{C})$. In this case $X = \mathbf{P}^1$. We let K denote the canonical bundle of X; K has degree -2. Let n be an integer. Fix once and for all an equivariant line bundle $\mathcal{O}(n)$ of degree n on $\mathbf{P}^1$; then $\mathcal{O}(n) \otimes K^{\otimes n+1}$ has degree $-n-2$. By Serre duality, the groups $\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(n))$ and $\mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(n) \otimes K^{\otimes n+1})$ are (non-canonically) dual, and therefore isomorphic as representations of G. Fix also a G-isomorphism ϕ between these vector spaces.

Now let $\mathcal{L}$ be any equivariant line bundle of degree n on $\mathbf{P}^1$. Then there exists an isomorphism $\mathcal{L} \simeq \mathcal{O}(n)$. The isomorphism ϕ gives rise to an isomorphism

$$\phi' : \mathrm{H}^0(\mathbf{P}^1, \mathcal{L}) \simeq \mathrm{H}^1(\mathbf{P}^1, \mathcal{L} \otimes K^{\otimes n+1})$$

Since $\mathcal{L}$ appears in the same way on both sides of this isomorphism, it follows that ϕ' depends only on ϕ, and not on the choice of isomorphism $\mathcal{L} \simeq \mathcal{O}(n)$. Thus ϕ' depends functorially on the pair $(\mathbf{P}^1, \mathcal{L})$. Descent theory then shows that ϕ' makes sense for families. That is, for any $\mathbf{P}^1$-bundle $\pi : E \to S$ and any line bundle $\mathcal{L}$ on E having degree n on fibers, we obtain a natural isomorphism

$$\pi_* \mathcal{L} \simeq R^1 \pi_* (\mathcal{L} \otimes K^{\otimes n+1})$$

If $n \geq -1$, these are the *only* direct images which do not vanish. Applying the (degenerate) Leray spectral sequence to this isomorphism, we obtain the following:

Theorem 3. *Let $\pi : E \to S$ be a $\mathbf{P}^1$-bundle with relative canonical bundle K, and let $\mathcal{L}$ be a line bundle on E with degree $n \geq -1$ on the fibers of S. There are natural isomorphisms*

$$\mathrm{H}^i(E, \mathcal{L}) \simeq \mathrm{H}^{i+1}(E, \mathcal{L} \otimes K^{\otimes n+1})$$

Using this fact, we can prove the full Borel-Weil-Bott theorem. In order to state the theorem, it is convenient to alter our conventions for indexing the equivariant line bundles on X. Let $\rho \in \Lambda$ denote half the sum of all the positive roots, and set $\mathcal{L}^\lambda = \mathcal{L}_{\rho-\lambda}$. Then Theorem 2 asserts that $\mathrm{H}^0(X, \mathcal{L}^\lambda)$ vanishes unless λ is dominant and regular, and is dual to the irreducible of highest weight $\lambda - \rho$ otherwise. The Borel-Weil-Bott theorem generalizes this to describe *all* the cohomology groups of equivariant line bundles on X.

Lemma 4. *Let α be a simple root, and suppose $\langle \alpha^\vee, \lambda \rangle \geq 0$. Then there is a canonical isomorphism*

$$\mathrm{H}^i(X, \mathcal{L}^\lambda) \simeq \mathrm{H}^{i+1}(X, \mathcal{L}^{w_\alpha(\lambda)})$$

where w_α denotes the simple reflection corresponding to α.

Proof. Let P be the minimal parabolic corresponding to the α. Then X is a $\mathbf{P}^1$-bundle over G/P and the restriction of $\mathcal{L}^\lambda$ to a fiber has degree $\langle \alpha^\vee, \lambda \rangle - 1 \geq -1$. The relative canonical bundle of the projection is given by $K = \mathcal{L}_\alpha$. Then

$$\mathcal{L}^{w_\alpha(\lambda)} = \mathcal{L}_{\rho - w_\alpha(\lambda)} = \mathcal{L}_{\rho-\lambda+\langle \alpha^\vee, \lambda \rangle \alpha} = \mathcal{L}^\lambda \otimes K^{\otimes \langle \alpha^\vee, \lambda \rangle}$$

and the desired isomorphism follows from Theorem 3. ∎

Let $\lambda \in \Lambda$. Some Weyl conjugate of $w(\lambda)$ of λ is dominant. If $w = w_1 \dots w_{l(w)}$ is a reduced expression for w as a product of simple reflections, corresponding to simple roots $\alpha_1, \dots \alpha_{l(w)}$, then for each i we have $\langle \alpha_i^\vee, w_{i+1} \dots w_{l(w)}(\lambda) \rangle \leq 0$. Applying the lemma repeatedly, we obtain isomorphisms

$$\mathrm{H}^{i+l(w)}(X, \mathcal{L}^\lambda) \simeq \mathrm{H}^i(X, \mathcal{L}^{w(\lambda)})$$

If λ is singular, then there is a simple root α such that $\langle \alpha^\vee, w(\lambda) \rangle = 0$, so that $w(\lambda)$ is invariant under reflection through α. The lemma yields isomorphisms

$$\mathrm{H}^i(X, \mathcal{L}^{w(\lambda)}) \simeq \mathrm{H}^{i+1}(X, \mathcal{L}^{w(\lambda)})$$

for each i. On the other hand, these groups vanish for $i = 0$ by Theorem 2. It follows that all the cohomology groups of $\mathcal{L}^\lambda$ vanish.

Now suppose λ is regular, and that $w(\lambda)$ is dominant. Then the isomorphism above shows that $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ vanishes for $i < l(w)$, and is dual to an irreducible representation of G of highest weight $w(\lambda) - \rho$ if $i = l(w)$. This proves most of the following result:

Theorem 5. (*Borel-Weil-Bott*) *Let $\lambda \in \Lambda$. If λ is not regular, then all the cohomology groups $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ vanish. If λ is regular, then there is a unique nonvanishing cohomology group $\mathrm{H}^{i(\lambda)}(X, \mathcal{L}^\lambda)$, where $i(\lambda)$ is the length of the unique element w of the Weyl group such that $w(\lambda)$ is dominant. In this case, $\mathrm{H}^{i(\lambda)}(X, \mathcal{L}^\lambda)$ is dual to an irreducible representation of G with highest weight $w(\lambda) - \rho$.*

Proof. All that remains to be shown is that for regular λ, $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ vanishes for $i > i(\lambda)$. Since $\mathcal{L}_{2\rho}$ is the canonical bundle of X, Serre duality implies that the vector space $\mathrm{H}^i(X, \mathcal{L}^\lambda)$ is dual to $\mathrm{H}^{n-i}(X, \mathcal{L}^{-\lambda})$, where n is the dimension of X. The desired result now follows, since $n - i < i(-\lambda)$. ∎

References

[1] Fulton, W. and J. Harris. *Representation Theory.* Springer-Verlag, 1991, pp. 392-393.