LECTURE 1: OVERVIEW

1. The Cohomology of Algebraic Varieties

Let Y be a smooth proper variety defined over a field K of characteristic zero, and let $\overline{K}$ be an algebraic closure of K. Then one has two different notions for the *cohomology* of Y:

- The *de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(Y)$, defined as the hypercohomology of Y with coefficients in its algebraic de Rham complex
$$\Omega^0_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \Omega^2_Y \xrightarrow{d} \cdots$$
This is graded-commutative algebra over the field of definition K, equipped with the *Hodge filtration*
$$\cdots \subseteq \mathrm{Fil}^2\,\mathrm{H}^*_{\mathrm{dR}}(Y) \subseteq \mathrm{Fil}^1\,\mathrm{H}^*_{\mathrm{dR}}(Y) \subseteq \mathrm{Fil}^0\,\mathrm{H}^*_{\mathrm{dR}}(Y) = \mathrm{H}^*_{\mathrm{dR}}(Y)$$
(where each $\mathrm{Fil}^i\,\mathrm{H}^*_{\mathrm{dR}}(Y)$ is given by the hypercohomology of the subcomplex $\Omega^{\geq i}_Y \subseteq \Omega^*_Y$).
- The *p-adic étale cohomology* $\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p)$, where $Y_{\overline{K}}$ denotes the fiber product $Y \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ and p is any prime number. This is a graded-commuative algebra over the field $\mathbf{Q}_p$ of p-adic rational numbers, equipped with a continuous action of the Galois group $\mathrm{Gal}(\overline{K}/K)$.

When $K = \mathbf{C}$ is the field of complex numbers, both of these invariants are determined by the singular cohomology of the underlying topological space $Y(\mathbf{C})$: more precisely, there are canonical isomorphisms
$$\mathrm{H}^*_{\mathrm{dR}}(Y) \simeq \mathrm{H}^*_{\mathrm{sing}}(Y(\mathbf{C}); \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$
$$\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \simeq \mathrm{H}^*_{\mathrm{sing}}(Y(\mathbf{C}); \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p\,.$$
In particular, if B is any ring equipped with homomorphisms $\mathbf{C} \hookrightarrow B \hookleftarrow \mathbf{Q}_p$, we have isomorphisms
$$\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_{\mathbf{C}} B \simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B.$$

One of the central objectives of p-adic Hodge theory is to understand the relationship between $\mathrm{H}^*_{\mathrm{dR}}(Y)$ and $\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p)$ in the case where K is a *p-adic field*. For the remainder of this lecture, we will assume that $K = W(k)[p^{-1}]$ for some perfect field k of characteristic p. (To fix ideas, the reader should feel free to take $k = \mathbf{F}_p$ so that $K = \mathbf{Q}_p$.) In this case, K is a complete with respect to a discrete valuation, with valuation ring $\mathcal{O}_K = W(k)$ and residue field k. This valuation extends uniquely to the algebraic closure $\overline{K}$, which has a *non-discrete* valuation subring $\mathcal{O}_{\overline{K}} \subseteq \overline{K}$. We let $\mathbf{C}_p$ denote the completion of $\overline{K}$ with respect to its valuation and $\mathcal{O}_{\mathbf{C}_p}$ its valuation ring (which is the p-adic completion of $\mathcal{O}_{\overline{K}}$).

2. Period Rings

In the case where Y arises as the generic fiber of a smooth proper scheme $\mathfrak{Y}$ over $\mathcal{O}_K = W(k)$, Grothendieck conjectured that the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(Y)$ and the p-adic étale cohomology $\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p)$ should "contain the same information," in the sense that either can be recovered from the other. To formulate a more precise version of this conjecture, Fontaine introduced certain *period rings* $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$. Let us begin by recalling some of the features of B_{dR}.

- (a) The ring B_{dR} is a field of characteristic zero. It is the fraction field of a complete discrete valuation ring B^+_{dR} having residue field $\mathbf{C}_p$.
- (b) The field B_{dR} is equipped with an action of the Galois group $\mathrm{Gal}(\overline{K}/K)$; this action preserves the valuation ring $B^+_{\mathrm{dR}} \subseteq B_{\mathrm{dR}}$ and is compatible with the usual action of $\mathrm{Gal}(\overline{K}/K)$ on the residue field $\mathbf{C}_p$.
- (c) The discrete valuation ring B^+_{dR} contains a uniformizing element t on which the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts by the formula $t^\sigma = \chi(\sigma)t$ where $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Q}_p^\times$ is the cyclotomic character.
- (d) The subfield of Galois-invariants $B_{\mathrm{dR}}^{\mathrm{Gal}(\overline{K}/K)}$ can be identified with K. More precisely, it is contained inB^+_{dR} and the composite map
$$B_{\mathrm{dR}}^{\mathrm{Gal}(\overline{K}/K)} \subseteq B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/tB^+_{\mathrm{dR}} \simeq \mathbf{C}_p$$
is a monomorphism with image $K \subseteq \mathbf{C}_p$.

Remark 1. The uniformizing element $t \in B^+_{\mathrm{dR}}$ is not quite canonical (in particular, it is not Galois-invariant); it depends on the choice of a compatible system of p^nth roots of unity in the field $\overline{K}$.

Warning 2. It follows from (a) that the field B^+_{dR} is abstractly isomorphic to a Laurent series field $\mathbf{C}_p((t))$. Beware that such an isomorphism is *highly* non-canonical. For example, such an isomorphism cannot be chosen in a $\mathrm{Gal}(\overline{K}/K)$-equivariant way.

Warning 3. The fields K and B_{dR} are both complete with respect to discrete valuations, and therefore inherit topologies. However, the inclusion $K \hookrightarrow B_{\mathrm{dR}}$ is not continuous for these topologies: the p-adic topology on K is unrelated to the t-adic topology on B_{dR}.

Fontaine then formulated the following conjecture:

Conjecture 4 (De Rham Comparison)**.** Let Y be a smooth proper algebraic variety over K. Then there is a canonical isomorphism of B_{dR}-vector spaces
$$\alpha_{\mathrm{dR}} : \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{dR}} \simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}.$$

Moreover:

(*i*) The isomorphism α_{dR} is $\mathrm{Gal}(\overline{K}/K)$-equivariant (where we equip $\mathrm{H}^*_{\mathrm{dR}}(Y)$ with the trivial action of $\mathrm{Gal}(\overline{K}/K)$).

(*ii*) The isomorphism α_{dR} respects filtrations (where we equip B_{dR} with the t-adic filtration $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B^+_{\mathrm{dR}}$ and $\mathrm{H}^*_{\mathrm{et}}(Y; \mathbf{Q}_p)$ with the trivial filration).

Remark 5. It follows from assertion (*ii*) that the isomorphism α_{dR} carries

$$\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B^+_{\mathrm{dR}} = \mathrm{Fil}^0 \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K \mathrm{Fil}^0 B_{\mathrm{dR}} \subseteq \mathrm{Fil}^0(\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{dR}})$$

to the subspace

$$\mathrm{Fil}^0(\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B^+_{\mathrm{dR}} \subseteq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}};$$

that is, it refines to a map of lattices

$$\alpha^+_{\mathrm{dR}} : \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B^+_{\mathrm{dR}} \to \mathrm{H}^*_{\mathrm{et}}(Y) \otimes_{\mathbf{Q}_p} B^+_{\mathrm{dR}}.$$

Beware that this map is generally not an isomorphism: assertion (*ii*) says more precisely that $\mathrm{H}^*_{\mathrm{et}}(Y) \otimes_{\mathbf{Q}_p} B^+_{\mathrm{dR}}$ is generated (as a B^+_{dR}-module) by the images of $t^{-i} \mathrm{Fil}^i \mathrm{H}^*_{\mathrm{dR}}(Y)$, where i varies.

It follows from part (*i*) of Conjecture 4 and property (*d*) above that the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(Y)$ can be recovered by the formula

$$\begin{aligned}
\mathrm{H}^*_{\mathrm{dR}}(Y) &\simeq \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K K \\
&\simeq \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{dR}}^{\mathrm{Gal}(\overline{K}/K)} \\
&\simeq (\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)} \\
&\simeq (\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)}.
\end{aligned}$$

To recover étale cohomology from de Rham cohomology, Fontaine also formulated a refinement of Conjecture 4. To this end, he introduced a subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ with the following features:

(b') The subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ is invariant under the action of $\mathrm{Gal}(\overline{K}/K)$ (and therefore inherits an action of $\mathrm{Gal}(\overline{K}/K)$).

(c') The subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ contains the element t and its inverse t^{-1}.

(d') The subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ contains K (which can therefore be identified with the ring of Galois invariants $B_{\mathrm{cris}}^{\mathrm{Gal}(\overline{K}/K)}$).

(*e*) The Frobenius automorphism of $K = W(k)[p^{-1}]$ extends to a ring homomorphism $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$, which we we also refer to as the Frobenius (beware that this homomorphism does not naturally extend to the larger ring B_{dR}).

Fontaine proposed the following variant of Conjecture 4:

Conjecture 6 (Crystalline Comparison)**.** Let $\mathfrak{Y}$ be a smooth proper scheme over the discrete valuation ring $\mathcal{O}_K = W(k)$ with generic fiber Y. Then α_{dR} restricts to an isomorphism

$$\alpha_{\mathrm{cris}} : \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}} \simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}$$

with the following features:

(i) The isomorphism α_{cris} is $\mathrm{Gal}(\overline{K}/K)$-equivariant (note that this is equivalent to the corresponding assertion of Conjecture 4).

(ii) The isomorphism α_{cris} respects filtrations, where we equip B_{cris} with the filtration

$$\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap \mathrm{Fil}^i B_{\mathrm{dR}} = t^i (B_{\mathrm{cris}} \cap B^+_{\mathrm{dR}})$$

induced by the filtration on B_{dR} (note that this implies the corresponding assertion of Conjecture 4.)

(iii) The isomorphism α_{cris} is Frobenius-equivariant. In other words, it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}} & \xrightarrow{\alpha_{\mathrm{cris}}} & \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}} \\ \downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi} \\ \mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}} & \xrightarrow{\alpha_{\mathrm{cris}}} & \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}, \end{array}$$

where $\varphi : \mathrm{H}^*_{\mathrm{dR}}(Y) \to \mathrm{H}^*_{\mathrm{dR}}(Y)$ is the isomorphism induced by the identification of $\mathrm{H}^*_{\mathrm{dR}}(Y)$ with the crystalline cohomology of the special fiber $\mathfrak{Y}_0 = \mathfrak{Y} \times_{\mathrm{Spec}\, W(k)} \mathrm{Spec}\, k$.

For any abelian group V equipped with an endomorphism φ, we let $V^{\varphi=1}$ denote the subgroup $\{v \in V : \varphi(v) = v\} \subseteq V$. One can show that the intersection $\mathrm{Fil}^0(B^{\varphi=1}_{\mathrm{cris}}) = B^{\varphi=1}_{\mathrm{cris}} \cap B^+_{\mathrm{dR}}$ coincides with $\mathbf{Q}_p$ (as a subring of B_{dR}). Using this fact, one can apply assertions (ii) and (iii) of Conjecture 6 to obtain an isomorphism

$$\begin{aligned} \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) &\simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p \\ &\simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathrm{Fil}^0 B^{\varphi=1}_{\mathrm{cris}} \\ &\simeq \mathrm{Fil}^0(\mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{\varphi=1} \\ &\simeq \mathrm{Fil}^0(\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}})^{\varphi=1}, \end{aligned}$$

which allows us to recover the étale cohomology of $Y_{\overline{K}}$ from the de Rham cohomology of Y (together with its Hodge filtration and the Frobenius automorphism arising from theory of crystalline cohomology).

Conjectures 4 and 6 were first proved by Faltings; there are now many proofs by many different methods.

3. The Fargues-Fontaine Curve

Let us now consider what happens to the isomorphism α_{cris} of Conjecture 6 if we take invariants with respect to Frobenius, but do *not* restrict our attention to filtration degree 0. In this case, we obtain a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism

$$\alpha : (\mathrm{H}^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}})^{\varphi=1} \simeq \mathrm{H}^*_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B^{\varphi=1}_{\mathrm{cris}}$$

of modules over the ring $B^{\varphi=1}_{\mathrm{cris}}$. This statement might a bit more palatable than Conjecture 6 by virtue of the following amazing fact:

Theorem 7. *The ring $B^{\varphi=1}_{\mathrm{cris}}$ is a principal ideal domain.*

Let E denote the fraction field of the domain $B^{\varphi=1}_{\mathrm{cris}}$ (which we can identify with a subfield of B_{dR}). Then E comes equipped with several discrete valuations:

- Every maximal ideal $\mathfrak{m} \subseteq B^{\varphi=1}_{\mathrm{cris}}$ determines a discrete valuation on E, whose valuation ring is the localization $(B^{\varphi=1}_{\mathrm{cris}})_{\mathfrak{m}}$.
- The inclusion $E \hookrightarrow B_{\mathrm{dR}}$ determines a discrete valuation on E, with valuation ring $E \cap B^+_{\mathrm{dR}}$.

Theorem 8. *There is a unique Dedekind scheme X with fraction field E having the following property: the local rings $\mathcal{O}_{X,x} \subseteq E$ at the closed points $x \in X$ are exactly the valuation subrings of E listed above.*

We will refer to the scheme X of Theorem 8 as the *Fargues-Fontaine curve.* Note that there is a unique closed point $\infty \in X$ having the property that $\mathcal{O}_{X,\infty} = E \cap B^+_{\mathrm{dR}}$. Then:

- The complement $X - \{\infty\}$ can be identified with the spectrum of the principal ideal domain $B^{\varphi=1}_{\mathrm{cris}}$.
- The inclusion $\mathcal{O}_{X,\infty} \hookrightarrow B^+_{\mathrm{dR}}$ exhibits B^+_{dR} as the completion of the discrete valuation ring $\mathcal{O}_{X,\infty}$. In other words, the formal completion of X at ∞ is given by $\mathrm{Spf}\, B^+_{\mathrm{dR}}$.

Remark 9. The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on the Fargues-Fontaine curve X, via its action on the period rings $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$.

We can describe the situation more informally by saying the Fargues-Fontaine curve is obtained by gluing the affine scheme $\mathrm{Spec}\, B^{\varphi=1}_{\mathrm{cris}}$ and the formal disk $\mathrm{Spec}\, B^+_{\mathrm{dR}}$ along the formal punctured disk $\mathrm{Spec}\, B_{\mathrm{dR}}$.

Note that a function $f \in E \subseteq B_{\mathrm{dR}}$ is regular on the Fargues-Fontaine curve X if and only if it is regular on $X - \{\infty\}$ (that is, it belongs to the subring $B^{\varphi=1}_{\mathrm{cris}} \subseteq E$) and regular at ∞ (that is, it belongs to B^+_{dR}). The equality $B^{\varphi=1}_{\mathrm{cris}} \cap B^+_{\mathrm{dR}} = \mathbf{Q}_p$ then yields the following calculation:

Theorem 10. *There is a canonical isomorphism $\mathrm{H}^0(X; \mathcal{O}_X) \simeq \mathbf{Q}_p$. In particular, X can be regarded as an $\mathbf{Q}_p$-scheme.*

Warning 11. The Fargues-Fontaine curve is not of finite type over $\mathbf{Q}_p$. For example, the residue field of X at the closed point $\infty \in X$ is given by $B^+_{\mathrm{dR}}/tB^+_{\mathrm{dR}} \simeq \mathbf{C}_p$, which is not a finite extension of $\mathbf{Q}_p$.

In spite of Warning 11, there are respects in which X behaves like a *complete* curve over an *algebraically closed* field:

Theorem 12. *Let f be an element of E. Then $\sum_{x \in X} v_x(f) = 0$, where v_x denotes the valuation on E determined by the point $x \in X$.*

Moreover, the Fargues-Fontaine curve X behaves like a curve of genus zero in the following sense:

Theorem 13. *The cohomology group $\mathrm{H}^1(X; \mathcal{O}_X)$ vanishes.*

Remark 14. Theorems 10 and 13 are together equivalent to the exactness of the sequence

$$0 \to \mathbf{Q}_p \to B^{\varphi=1}_{\mathrm{cris}} \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

This is sometimes called the *fundamental exact sequence* of p-adic Hodge theory.

4. Vector Bundles on the Fargues-Fontaine Curve

We now discuss the classification of vector bundles on the Fargues-Fontaine curve X. Using Theorem 12, one can show that every coherent sheaf $\mathcal{E}$ on X has a well-defined *degree* $\deg(\mathcal{E})$, which is uniquely determined by the following requirements:

- For every exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we have $\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'')$.
- For any closed point $x \in X$ and any integer n, the line bundle $\mathcal{O}_X(nx)$ has degree n.

If $\mathcal{E}$ is a vector bundle, we define the *slope* of $\mathcal{E}$ to be the ratio $\frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$. We say that $\mathcal{E}$ is *semistable* if, for every subsheaf $\mathcal{E}' \subseteq \mathcal{E}$, we have $\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\mathcal{E})$. We then have the following

Theorem 15. *Let $\mathcal{E}$ be a vector bundle on X. Then there is a unique filtration*

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E},$$

where each quotient $\mathcal{E}_i / \mathcal{E}_{i-1}$ is semistable, and their slopes satisfy the inequalities

$$\mathrm{slope}(\mathcal{E}_1/\mathcal{E}_0) > \mathrm{slope}(\mathcal{E}_2/\mathcal{E}_1) > \mathrm{slope}(\mathcal{E}_3/\mathcal{E}_2) > \cdots > \mathrm{slope}(\mathcal{E}_n/\mathcal{E}_{n-1}).$$

We will refer to this as the Harder-Narasimhan filtration. Moreover, this filtration is (non-canonically) split.

Remark 16. The splitting of the Harder-Narasimhan filtration of vector bundles on X lends further support to the heuristic idea that X behaves like a curve of genus zero: recall that every vector bundle $\mathcal{E}$ on the projective line $\mathbf{P}^1$ splits as a direct sum $\mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \cdots \oplus \mathcal{O}(n_r)$. where the line bundles $\mathcal{O}(n_i)$ are semistable of slope n_i.

Using Theorem 15, the problem of classifying vector bundles on X is reduced to the problem of classifying semistable vector bundles of slope λ, where λ is a rational number. Here the situation is more interesting than the analogous situation for $\mathbf{P}^1$, because non-integral slopes appear:

Theorem 17. *Let λ be a rational number and let $\mathcal{C}$ be the category whose objects are semistable vector bundles on X of slope λ. Then $\mathcal{C}$ is an abelian category with a unique simple object $\mathcal{O}(\lambda)$, whose endomorphism algebra can be identified with a central divison algebra over $\mathbf{Q}_p$ whose Hasse invariant is given by the image of λ in $\mathbf{Q}/\mathbf{Z} \simeq \mathrm{Br}(\mathbf{Q}_p)$.*

Understanding Theorems 15 and 17 will be one of the main goals of this seminar. Note that although these theorems are easy to state, the proofs will take us through some interesting mathematics: namely, period morphisms on moduli spaces of p-divisible groups. Moreover, these theorems have quite nontrivial consequences: for example, they can be used to give a conceptual proof of the *Colmez-Fontaine theorem* ("weakly admissible implies admissible").

5. The Archimedean Case: The Twistor Line

Before going further, let us mention that many of the Fargues-Fontaine curve has an analogue in archimedean geometry, which was introduced by Carlos Simpson. Let $\mathbf{P}^1_{\mathbf{C}}$ denote the projective line over the field $\mathbf{C}$ of complex numbers. Then $\mathbf{P}^1_{\mathbf{C}}$ is equipped with a conjugate-linear involution, given by the antipodal map $z \mapsto -1/\overline{z}$. This involution supplies descent data for a real form of $\mathbf{P}^1_{\mathbf{C}}$, which we will denote by $\mathbf{P}^1_{\mathrm{tw}}$. We will refer to $\mathbf{P}^1_{\mathrm{tw}}$ as the *twistor line*. It has the following features:

- The twistor line $\mathbf{P}^1_{\mathrm{tw}}$ is an algebraic curve of genus zero over $\mathbf{R}$. However, it has no $\mathbf{R}$-valued points: the residue field of $\mathbf{P}^1_{\mathrm{tw}}$ at every closed point is (noncanonically) isomorphic to $\mathbf{C}$. (In fact, $\mathbf{P}^1_{\mathrm{tw}}$ is characterized up to isomorphism by these properties.)
- The map $\operatorname{Spec} \mathbf{C} \simeq \{\infty\} \hookrightarrow \mathbf{P}^1_{\mathbf{C}}$ induces a closed immersion of $\mathbf{R}$-schemes $i : \operatorname{Spec} \mathbf{C} \hookrightarrow \mathbf{P}^1_{\mathrm{tw}}$ which picks out a "point at infinity". Let us identify the formal completion $\mathbf{P}^1_{\mathrm{tw}}$ along i with the formal scheme $\operatorname{Spf} \mathbf{C}[[\lambda]]$.
- Automorphisms of $\mathbf{P}^1_{\mathrm{tw}}$ that preserve the closed immersion i are parametrized by an algebraic group G over $\mathbf{R}$, which is a non-split form of the multiplicative group $\mathbf{G}_m$ (note that $\mathbf{G}_m$ can be identified with the group of automorphisms of the projective line which fix the points 0 and ∞). The

action of G on $\mathbf{P}^1_{\mathrm{tw}}$ restricts to a free action of G on $\mathbf{P}^1_{\mathrm{tw}} - \{\infty\}$: that is, it exhibits $\mathbf{P}^1_{\mathrm{tw}} - \{\infty\}$ as a (nontrivial!) G-torsor.

- The evident equivalence

$$\begin{array}{c}\{G\text{-equivariant vector bundles on } \mathbf{P}^1_{\mathrm{tw}} - \{\infty\}\} \\ \downarrow \sim \\ \{\text{Finite-dimensional vector spaces over } \mathbf{R}\}\end{array}$$

can be upgraded to an equivalence of categories

$$\begin{array}{c}\{G\text{-equivariant vector bundles on } \mathbf{P}^1_{\mathrm{tw}}, \text{ semistable of slope } \tfrac{n}{2}\} \\ \downarrow \sim \\ \{\text{Pure Hodge structures of weight } n\}.\end{array}$$

Here we measure slopes using the procedure of §4 (so they are half as large as the slopes of the corresponding vector bundles on $\mathbf{P}^1_{\mathbf{C}}$).

We have the following table of analogies:

Archimedean Geometry	Nonarchimedean Geometry
The Local Field $\mathbf{R}$	The Local Field $\mathbf{Q}_p$
Algebraic Closure $\mathbf{C}$	Completed Algebraic Closure $\mathbf{C}_p$
Twistor Line $\mathbf{P}^1_{\mathrm{tw}}$	Fargues-Fontaine Curve X
$i : \operatorname{Spec} \mathbf{C} \hookrightarrow \mathbf{P}^1_{\mathrm{tw}}$	$\operatorname{Spec} \mathbf{C}_p = \{\infty\} \hookrightarrow X$
Power series ring $\mathbf{C}[[\lambda]]$	Period Ring B_{dR}
Action of G on $\mathbf{P}^1_{\mathrm{tw}}$	Action of $\mathrm{Gal}(\overline{K}/K)$ on X
Pure Hodge structures	$\mathrm{Gal}(\overline{K}/K)$-equivariant (?semistable?) bundles on X

Let us explain, in this language, how to associate to a smooth projective variety Y over $\mathbf{C}$ the associated pure Hodge structure on the cohomology group $\mathrm{H}^n_{\mathrm{sing}}(Y(\mathbf{C}); \mathbf{C})$. First, one considers the trivial vector bundle

$$\mathcal{E} = \mathrm{H}^n_{\mathrm{sing}}(Y(\mathbf{C}); \mathbf{R}) \otimes_{\mathbf{R}} \mathcal{O}_{\mathbf{P}^1_{\mathrm{tw}}}$$

on the twistor line $\mathbf{P}^1_{\mathrm{tw}}$. The global sections of $\mathcal{E}$ over the formal disk $\operatorname{Spf} \mathbf{C}[[\lambda]]$ can be identified with the nth hypercohomology group of the de Rham complex

$$\Omega^0_Y[[\lambda]] \xrightarrow{d} \Omega^1_Y[[\lambda]] \xrightarrow{d} \Omega^2_Y[[\lambda]] \to \cdots.$$

We can then construct a new vector bundle which is isomorphic to $\mathcal{E}$ away from the image of i, and whose global sections over the formal completion $\operatorname{Spf} \mathbf{C}[[\lambda]]$ are given by the nth hypercohomology of the larger complex

$$\Omega^0_Y[[\lambda]] \xrightarrow{d} \lambda^{-1}\Omega^1_Y[[\lambda]] \xrightarrow{d} \lambda^{-2}\Omega^2_Y[[\lambda]] \to \cdots.$$

This construction yields a G-equivariant vector bundle $\mathcal{F}$ on $\mathbf{P}^1_{\mathrm{tw}}$ which is semistable of slope $n/2$ (according to the convention above), whose fiber at infinity is given by the Dolbeault cohomology group $\bigoplus_{p+q=n} \mathrm{H}^p(Y; \Omega^q_Y)$, and whose fiber at every other point is given by the singular cohomology group $\mathrm{H}^n_{\mathrm{sing}}(Y(\mathbf{C}); \mathbf{C})$.

Remark 18. The well-definedness of the vector bundle $\mathcal{F}$ above follows from the degeneration of the Hodge-to-de-Rham spectral sequence for Y, and the fact that it is semistable of slope $n/2$ is a reformulation of the fact that the Hodge filtration is split by its complex conjugate.

Remark 19. In the preceding situation, the vector bundle $\mathcal{E}$ can be recovered from $\mathcal{F}$: it is the largest subsheaf of $\mathcal{F}$ which is generated by G-invariant global sections in a formal neighborhood of the point ∞.

6. Modifications of Vector Bundles on X

Let us now suppose that Y is a smooth proper variety over $K = W(k)[p^{-1}]$. In this case, we can use the de Rham comparison isomorphism (Conjecture 4) to perform an analogue of the preceding construction. More precisely, for every nonnegative integer n, we can construct a vector bundle $\mathcal{E}$ on the Fargues-Fontaine curve X by the following prescription:

- Over the open set $X - \{\infty\} \simeq \operatorname{Spec} B_{\mathrm{cris}}^{\varphi=1}$, we take $\mathcal{E}$ to be the trivial vector bundle associated to the $\mathbf{Q}_p$-vector space $\mathrm{H}^n_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p)$.
- Over the formal disk $\widehat{X}_\infty = \operatorname{Spec} B^+_{\mathrm{dR}}$, we take $\mathcal{E}$ to be the trivial vector bundle associated to the K-vector space $\mathrm{H}^n_{\mathrm{dR}}(Y)$.
- On the overlap $\operatorname{Spec} B_{\mathrm{dR}} = (X - \{\infty\}) \times_X \operatorname{Spec} B^+_{\mathrm{dR}}$, we glue the vector bundles $\mathcal{E}|_{X-\{\infty\}}$ and $\mathcal{E}|_{\operatorname{Spec} B^+_{\mathrm{dR}}}$ using the de Rham comparison isomorphism

$$\mathrm{H}^n_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq \mathrm{H}^n_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{dR}}.$$

Recall that the de Rham comparison isomorphism induces a map of lattices

$$\mathrm{H}^n_{\mathrm{dR}}(Y) \otimes_K B^+_{\mathrm{dR}} \hookrightarrow \mathrm{H}^n_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B^+_{\mathrm{dR}},$$

which determines a map from $\mathcal{E}$ to the trivial vector bundle

$$\mathcal{F} = \mathrm{H}^n_{\mathrm{et}}(Y_{\overline{K}}; \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathcal{O}_X\,.$$

This map is an isomorphism over the open set $X - \{\infty\}$, and therefore exhibits $\mathcal{E}$ as a *modification of* $\mathcal{F}$ *at* ∞.

Remark 20. As in the archimedean case, we can recover $\mathcal{E}$ from $\mathcal{F}$: it is the largest subsheaf of $\mathcal{F}$ which is generated by $\mathrm{Gal}(\overline{K}/K)$-equivariant global sections in a formal neighborhood of ∞ (this is a translation of the statement that the de Rham comparison isomorphism preserves filtrations).

Warning 21. Note some differences between the archimedean and nonarchimedean constructions:

- In both cases, we obtain a modification of vector bundles $\mathcal{E} \subseteq \mathcal{F}$. In the archimedean case, the vector bundle $\mathcal{E}$ is trivial (associated to the singular cohomology $\mathrm{H}^n_{\mathrm{sing}}(Y;\mathbf{R})$), and the vector bundle $\mathcal{F}$ has slope $n/2$. In the non-archimedean case it is $\mathcal{F}$ which is trivial (associated to the étale cohomology $\mathrm{H}^n_{\mathrm{et}}(Y_{\overline{K}};\mathbf{Q}_p)$), and $\mathcal{E}$ has slope $-n/2$.
- The vector bundle $\mathcal{E}$ constructed above is generally not semistable. For example, if $n = 1$ and Y is an elliptic curve over K, then $\mathcal{E}$ is semistable if and only if Y has supersingular reduction.

Remark 22. Using work of Scholze, one can extend the preceding construction to the setting where Y is a smooth proper rigid analytic variety over $\mathbf{C}_p$ (however, in the general case, the identification of $\mathcal{E}$ with de Rham cohomology can be carried out only at the point $\infty \in X$, rather than in a formal neighborhood of ∞).

7. Tate Duality

Let C be a smooth proper algebraic curve defined over an algebraically closed field F of characteristic zero. Then the étale cohomology of C satisfies Poincaré duality. More precisely, for every positive integer n, there is a *trace map*

$$e_C : \mathrm{H}^2_{\mathrm{et}}(C;\mu_n) \xrightarrow{\sim} \mathbf{Z}/n\,\mathbf{Z}$$

with the following property: for every constructible locally constant sheaf of $(\mathbf{Z}/n\,\mathbf{Z})$-modules $\mathcal{F}$ on C having dual $\mathcal{G} = \mathrm{Hom}(\mathcal{F},\mu_n)$, the pairing

$$\mathrm{H}^*_{\mathrm{et}}(C;\mathcal{F}) \times \mathrm{H}^{2-*}_{\mathrm{et}}(C;\mathcal{G}) \to \mathrm{H}^2_{\mathrm{et}}(C;\mu_n) \xrightarrow{e_C} \mathbf{Z}/n\,\mathbf{Z}$$

is perfect.

Tate proved that the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is a (profinite) *Poincare duality* group of dimension 2. More precisely, there are isomorphisms $e_{\mathbf{Q}_p} : \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p;\mu_n) \xrightarrow{\sim} \mathbf{Z}/n\,\mathbf{Z}$ having the same property. In this sense, the scheme $\mathrm{Spec}\,\mathbf{Q}_p$ behaves like a smooth proper curve over an algebraically closed field. However, there are other respects in which it behaves differently. Note that on any $\mathbf{Z}[\frac{1}{n}]$-scheme Z, we have an exact sequence of étale sheaves

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0.$$

This gives rise to a long exact sequence

$$\mathrm{Pic}(Z) \xrightarrow{n} \mathrm{Pic}(Z) \to \mathrm{H}^2_{\mathrm{et}}(Z;\mu_n) \to \mathrm{Br}(Z) \xrightarrow{n} \mathrm{Br}(Z)$$

(here $\mathrm{Br}(Z)$ denotes the *cohomological Brauer group* $\mathrm{H}^2_{\mathrm{et}}(Z;\mathbf{G}_m)$). Note that:

- When $Z = C$ is a smooth proper curve over an algebraically closed field, the Brauer group $\mathrm{Br}(Z)$ vanishes. The trace map can therefore be viewed as an isomorphism $\mathrm{Pic}(Z)/n\,\mathrm{Pic}(Z) \simeq \mathbf{Z}/n\mathbf{Z}$, which assigns to each line bundle $\mathcal{L}$ on Z its degree (taken modulo n).
- When $Z = \mathrm{Spec}\,\mathbf{Q}_p$, the Picard group $\mathrm{Pic}(Z)$ vanishes. The nontriviality of $\mathrm{H}^2_{\mathrm{et}}(Z;\mu_n)$ instead comes from torsion in the Brauer group $\mathrm{Br}(Z)$.

The theory of the Fargues-Fontaine curve X allows us to understand Tate duality in a more geometric way: that is, to relate it to a duality phenomenon which behaves more like the duality of a smooth proper curve over an algebraically closed field.

Theorem 23. (1) *The projection map $\pi : X \to \mathrm{Spec}\,\mathbf{Q}_p$ induces an isomorphism of étale fundamental groups $\pi_1^{\mathrm{et}}(X) \to \pi_1(\mathrm{Spec}\,\mathbf{Q}_p) = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.*

(2) *The construction $\mathcal{F} \mapsto \pi^*\mathcal{F}$ induces an equivalence from the category of torsion étale sheaves on $\mathrm{Spec}\,\mathbf{Q}_p$ to the category of torsion étale sheaves on X.*

(3) *Let $\mathcal{F}$ be a torsion étale sheaf on $\mathrm{Spec}\,\mathbf{Q}_p$. Then the canonical map $\mathrm{H}^*_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p;\mathcal{F}) \to \mathrm{H}^*_{\mathrm{et}}(X;\pi^*\mathcal{F})$ is an isomorphism.*

The theory of Tate duality for $\mathrm{Spec}\,\mathbf{Q}_p$ then translates to a similar statement for the Fargues-Fontaine curve X. Moreover, pullback along π determines a map of long exact sequences

$$\begin{array}{ccccccccc}
\mathrm{Pic}(\mathbf{Q}_p) & \xrightarrow{n} & \mathrm{Pic}(\mathbf{Q}_p) & \longrightarrow & \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p;\mu_n) & \longrightarrow & \mathrm{Br}(\mathbf{Q}_p) & \xrightarrow{n} & \mathrm{Br}(\mathbf{Q}_p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(X) & \xrightarrow{n} & \mathrm{Pic}(X) & \longrightarrow & \mathrm{H}^2_{\mathrm{et}}(X;\mu_n) & \longrightarrow & \mathrm{Br}(X) & \xrightarrow{n} & \mathrm{Br}(X).
\end{array}$$

But the lower line of this diagram behaves differently from the upper line: one can show that the map $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ is an isomorphism (this is equivalent to the assertion that $B_{\mathrm{cris}}^{\varphi=1}$ is a principal ideal domain), and that the Brauer group $\mathrm{Br}(X)$ vanishes. Consequently, Tate duality becomes more "geometric" when reformulated in terms of X. Moreover, the trace map $\mathrm{H}^2_{\mathrm{et}}(X;\mu_n) \to \mathbf{Z}/n\mathbf{Z}$ remains nontrivial when restricted to the relative cohomology $\mathrm{H}^2_{\mathrm{et}}(X, X-\{x\};\mu_n)$ for each closed point $x \in X$ (reflecting the idea that the duality phenomenon is occurring *locally* on X).

Remark 24. The vanishing of the Brauer group $\mathrm{Br}(X)$ implies the vanishing of the pullback map $\pi^* : \mathrm{Br}(\mathbf{Q}_p) \to \mathrm{Br}(X)$. Concretely, this tells us that every central division algebra D over $\mathbf{Q}_p$ has the property that the Azumaya algebra $\pi^* D$ arises as $\mathrm{End}(\mathcal{E})$, for some vector bundle $\mathcal{E}$ on X. Here we take $\mathcal{E} = \mathcal{O}(\lambda)$, where $\lambda \in \mathbf{Q}$ is a lift of the Hasse invariant of D (see Theorem 17).

8. Tilting and Untilting

By construction, the Fargues-Fontaine curve X comes equipped with a distinguished point $\infty \in X$, with complete local ring B_{dR}^+ and residue field $\mathbf{C}_p$. We now ask the question: what are the other (closed) points of X? To answer this, we first need to review a few definitions.

Definition 25. Let C be a field which is complete with respect to a (rank 1) valuation, having valuation ring $\mathcal{O}_C$ whose residue field is of characteristic p. We say that C is a *perfectoid field* if the valuation on C is nondiscrete and the Frobenius map

$$\varphi : \mathcal{O}_C / p \to \mathcal{O}_C / p$$

is surjective.

Example 26. A complete valued field C of characteristic p is perfectoid if and only if it is perfect: that is, if and only if the Frobenius map $\varphi : C \to C$ is bijective.

Example 27. Let C be an algebraically closed field which is complete with respect to a (rank 1)-valuation, whose residue field has characteristic p. Then C is perfectoid: in fact, the map $x \mapsto x^p$ is already surjective on $\mathcal{O}_C$, even before dividing out by the ideal (p).

To any perfectoid field C, one can associate a perfectoid field $C^\flat$ of characteristic p, called the *tilt* of C. It is characterized by the existence of an isomorphism

$$\mathcal{O}_{C^\flat} \simeq \varprojlim(\cdots \to \mathcal{O}_C / p \xrightarrow{\varphi} \mathcal{O}_C / p \xrightarrow{\varphi} \mathcal{O}_C / p \xrightarrow{\varphi} \mathcal{O}_C / p).$$

One can show that the category of finite extension fields of C is equivalent to the category of finite extension fields of $C^\flat$ (via tilting), so that the absolute Galois groups of C and $C^\flat$ are isomorphic. In particular, the field C is algebraically closed if and only if the field $C^\flat$ is algebraically closed.

Definition 28. Let F be a perfectoid field of characteristic p. An *untilt* of F is a perfectoid field C together with a (continuous) isomorphism $\iota : C^\flat \simeq F$. We let $\mathrm{Un}(F)$ denote the collection of all isomorphism classes of untilts (C, ι), where C is a perfectoid field of characteristic zero.

Note that if F is a perfectoid field of characteristic p, then the set $\mathrm{Un}(F)$ is acted on by the automorphism group of F. In particular, the Frobenius map φ induces an automorphism of $\mathrm{Un}(F)$.

Theorem 29. *There is a canonical bijection*

$$\{\textit{Closed points of the Fargues-Fontaine Curve } X\} \simeq \mathrm{Un}(\mathbf{C}_p^\flat)/(\varphi^{\mathbf{Z}}).$$

This bijection carries the point $\infty \in X$ *to (the Frobenius orbit of) the untilt* $(\mathbf{C}_p, \mathrm{id}_{\mathbf{C}_p^\flat})$.

Remark 30. The construction of the Fargues-Fontaine curve X sketched in this lecture depends on the choice of a perfect field k of characteristic p and an algebraic closure $\overline{K}$ of the field $K = W(k)[p^{-1}]$. However, Theorem 29 suggests that it depends only on the field $\mathbf{C}_p^\flat$. This suggestion is correct: moreover, one can associate an analogue $X_{C^\flat}$ of the Fargues-Fontaine curve to *any* perfectoid field $C^\flat$ of characteristic p (which need not be algebraically closed). Beware however that this construction does not come equipped with a canonical "point at infinity": this depends on a choice of untilt of $C^\flat$.

Warning 31. Many of the results we have stated in this lecture for the Fargues-Fontaine curve $X = X_{\mathbf{C}_p^\flat}$ depend on the fact that $\mathbf{C}_p^\flat$ is algebraically closed, and need to be modified when we consider perfectoid fields which are not algebraically closed. For example, if $C^\flat$ is not algebraically closed, then the closed points of $X_{C^\flat}$ can be identified with (Frobenius orbits of) untilts of *finite extensions* of $C^\flat$, rather than $C^\flat$ itself.

Remark 32. Scholze has introduced a variant of the Fargues-Fontaine curve $X_{C^\flat}$, where the perfectoid field $C^\flat$ is replaced by a *perfectoid space* S of characteristic p. This variant is not a $\mathbf{Q}_p$-scheme: instead, it is an *adic space over* $\mathbf{Q}_p$, which we will denote by X_S^{ad} and refer to as the *adic Fargues-Fontaine curve*. In the special case where S is the adic spectrum of a perfectoid field $C^\flat$, it is useful to think of X_S^{ad} as an "analytification" of the scheme $X_{C^\flat}$: for example, there is a GAGA theorem relating coherent sheaves on $X_{C^\flat}$ with coherent sheaves on X_S^{ad}. In these lectures, we will confine our attention to the schematic version of the Fargues-Fontaine curve.

Let us close this lecture with a few remarks about why one might want to have a "geometric" classification for untilts of a perfectoid field. We first make some general remarks about the arithmetic of global fields. Generally speaking, function fields are much easier to study than number fields because of their close connection with algebraic geometry: if F is the function field of an algebraic curve C over a finite field $\mathbf{F}_q$, then one can use the geometry of C to understand F. If F is a number field with ring of integers $\mathcal{O}_F \subseteq F$, one can try to regard the spectrum $\operatorname{Spec} \mathcal{O}_F$ as a replacement for the algebraic curve C, but it is a poor substitute. In the function field case, we can consider not only C itself, but also the product $C \times_{\operatorname{Spec} \mathbf{F}_q} C$. By contrast, the product $\operatorname{Spec} \mathbf{Z} \times \operatorname{Spec} \mathbf{Z}$, when formed in the category of schemes, is simply $\operatorname{Spec} \mathbf{Z}$ again: every scheme T admits a unique structural morphism $T \to \operatorname{Spec} \mathbf{Z}$.

Scholze proposed a beautiful idea for how one might salvage the idea of taking fiber products like $\operatorname{Spec}\mathbf{Z} \times \operatorname{Spec}\mathbf{Z}$, at least after p-adic completion:

- Rather than working in the category of schemes, one works in the setting of perfectoid spaces S of characteristic p.
- The notion of a structural morphism $S \to \operatorname{Spf}\mathbf{Z}_p$ is replaced by the notion of choosing an untilt of S (note that an untilt is an object in the category of adic spaces over formal spectrum $\operatorname{Spf}\mathbf{Z}_p$).

In this setting, can attach a nontrivial meaning to the product $\operatorname{Spf}\mathbf{Z}_p \times \operatorname{Spf}\mathbf{Z}_p$: it parametrizes the choices of *pairs* of untilts of S. To work effectively with this notion, one needs to understand how untilts are parametrized, and here the Fargues-Fontaine curve (in its adic incarnation) plays a central role.

Based on these ideas, Fargues and Scholze have proposed a program for importing geometric ideas into the arithmetic of the local Langlands program. The philosophy of their approach can be roughly summarized by the heuristic slogan

$$\{\text{Local Langlands for } \mathbf{Q}_p\} \simeq \{\text{Geometric Langlands for the Fargues-Fontaine Curve}\}.$$

Our goal in this seminar will be to understand the classification of vector bundles (and possible principal bundles for other groups) on the Fargues-Fontaine curve, which is fundamental to making sense of the right hand side.