Constructible Sheaves (Lecture 18)

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In this lecture, we describe the theory of *constructible sheaves* on a polyhedron. First, we summarize a few ideas we will need from piecewise linear topology.

Definition 1. Let K be a polyhedron equipped with a triangulation T. For every simplex $\sigma \in T$, we define $\mathrm{lk}(\sigma)$ to be the union of those simplices $\tau \in T$ such that $\tau \cap \sigma = \emptyset$ and such that $\tau \cup \sigma$ is contained in a simplex of T. The set $\mathrm{lk}(\sigma)$ is called the *link* of σ.

We will need the following:

Fact 2. *Let M be a piecewise linear n manifold with boundary equipped with a trianguation T. If σ is a k-simplex of M which does not belong to ∂M, then $\mathrm{lk}_\sigma(M)$ is PL homeomorphic to a sphere $S^{n-1-k} \simeq \partial \Delta^{n-k}$. If $\sigma \subseteq \partial M$, then $\mathrm{lk}(\sigma)$ is PL homeomorphic to a disk D^{n-1-k} (with ∂D^{n-1-k} being the link of σ in ∂M).*

Definition 3. Let X be a polyhedron equipped with a triangulation T. We regard T as a partially ordered set (with respect to inclusions of simplices). Let $\mathcal{C}$ be an ∞-category. We will let $\mathrm{Shv}_T(X; \mathcal{C})$ denote the ∞-category of functors from T to $\mathcal{C}$. We will refer to $\mathrm{Shv}_T(X; \mathcal{C})$ as the *∞-category of $\mathcal{C}$-valued T-constructible sheaves on X.*

For the remainder of this lecture, we will fix a polyhedron X and a stable ∞-category $\mathcal{C}$. Given a triangulation T of X, we let say that an object $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$ is *compactly supported* if $\mathcal{F}(\tau) = 0$ for all but finitely many simplices $\tau \in T$. Let $\mathrm{Shv}^c_T(X; \mathcal{C})$ denote the full subcategory of $\mathrm{Shv}_T(K; \mathcal{C})$ spanned by the compactly supported objects.

Example 4. Let $\tau_0 \in T$ and $C \in \mathcal{C}$, and define $\mathcal{F}^{\tau_0, C} : T \to \mathcal{C}$ by the formula

$$\mathcal{F}^{\tau_0, C}(\tau) = \begin{cases} C & \text{if } \tau_0 \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}^{\tau_0, C}$ is a compactly supported T-constructible sheaf on X. It has the following universal property: for every $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, there is a canonical homotopy equivalence

$$\mathrm{Mor}_{\mathrm{Shv}_T(X; \mathcal{C})}(\mathcal{F}^{\tau_0, C}, \mathcal{F}) \simeq \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{F}(\tau_0)).$$

Remark 5. The stable ∞-category $\mathrm{Shv}^c_T(X; \mathcal{C})$ is generated (under the formation of fibers and cofibers) by objects of the form $\mathcal{F}^{\tau_0, C}$. To prove this, let $\mathcal{F}$ be an arbitrary compactly supported T-constructible sheaf on X; we will show that $\mathcal{F}$ belongs to the smallest stable subcategory of $\mathrm{Shv}^c_T(X; \mathcal{C})$ containing the sheaves $\mathcal{F}^{\tau_0, C}$. Since $\mathcal{F}$ is compactly supported, there exists a finite upward-closed subset $T_0 \subseteq T$ such that $\mathcal{F}(\tau) = 0$ for $\tau \notin T_0$. We proceed by induction on the size of T_0. If T_0 is empty, then $\mathcal{F} \simeq 0$ and there is nothing to prove. Otherwise, choose a minimal element $\tau_0 \in T_0$, let $C = \mathcal{F}(\tau_0)$, and form a cofiber sequence

$$\mathcal{F}^{\tau_0, C} \to \mathcal{F} \to \mathcal{F}'.$$

The desired result then follows by applying the inductive hypothesis to $\mathcal{F}'$.

Definition 6. Let T be a trianguation of X. We define the *global sections functor* $\Gamma : \mathrm{Shv}^{\mathcal{C}}_T(X;\mathcal{C})$ by the formula

$$\Gamma(\mathcal{F}) = \varprojlim_{\tau \in T} \mathcal{F}(\tau).$$

Example 7. Suppose that the polyhedron X is finite (or that $\mathcal{C}$ admits infinite limits). Let $C \in \mathcal{C}$ be an object and let $X_0 \subseteq X$ be a closed subset which is of simplices belonging to T. Define $\mathcal{F}_{X_0}$ by the formula

$$\mathcal{F}_{X_0}(\tau) = \begin{cases} C & \text{if } \tau \subseteq X_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma(\mathcal{F}_{X_0})$ can be identified with the mapping object C^{X_0} (characterized by the universal property $\mathrm{Map}_{\mathcal{C}}(C', C^{X_0}) = \mathrm{Map}_{\mathcal{C}}(C', C)^{X_0}$).

Suppose now that τ is a simplex of T. The *open star* of τ is the union of the interiors of those simplices which contain τ (this is an open subset of X). Let X_0 be the complement of the open star of τ. We have a fiber sequence of functors

$$\mathcal{F}^{\tau,C} \to \mathcal{F}_X \to \mathcal{F}_{X_0}.$$

It follows that $\Gamma(\mathcal{F}^{\tau,C})$ can be identified with the mapping object $C^{(X,X_0)}$ (which is characterized by the universal property that $\mathrm{Map}_{\mathcal{C}}(C', C^{(X,X_0)})$ is the homotopy fiber of the map $\mathrm{Map}_{\mathcal{C}}(C',C)^X \to \mathrm{Map}_{\mathcal{C}}(C',C)^{X_0}$). It is not difficult to see that X_0 is a deformation retract of $X - \{x\}$, where x is any point belonging to the interior of τ. We can therefore write $\Gamma(\mathcal{F}^{\tau,C}) = C^{(X,X-\{x\})}$.

We now study the dependence of the ∞-category $\mathrm{Shv}_T(X;\mathcal{C})$ on the choice of triangulation T. Suppose that S is a triangulation of K which refines T. Then every simplex $\sigma \in S$ is contained in a simplex $\tau \in T$. We will denote the smallest such simplex by $i(\sigma)$. We regard i as a map of partially ordered sets $S \to T$, which induces (by composition) a functor $i^* : \mathrm{Shv}_T(X;\mathcal{C}) \to \mathrm{Shv}_S(X;\mathcal{C})$.

Proposition 8. *In the situation above, the functor i^* is fully faithful.*

Let us sketch the proof of Proposition 8. The pullback functor i^* has a left adjoint $i_+ : \mathrm{Shv}_S(X;\mathcal{C}) \to \mathrm{Shv}_T(X;\mathcal{C})$, given by left Kan extension along i. Concretely, this functor can be described by the formula

$$(i_+ \mathcal{F})(\tau) = \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{F}(\sigma),$$

where $\mathcal{F} \in \mathrm{Shv}_S(X;\mathcal{C})$. To prove Proposition 8, we must show that for every $\mathcal{G} \in \mathrm{Shv}_T(X;\mathcal{C})$, the counit map $i_+ i^* \mathcal{G} \to \mathcal{G}$ is an equivalence. Evaluating at a simplex $\tau \in T$, we are required to prove that $\mathcal{G}(\tau)$ is given by the colimit $\varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(i(\sigma))$. In other words, we wish to show that the canonical map

$$\theta : \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(i(\sigma)) \to \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(\tau)$$

is an equivalence (the right hand side is given by $\mathcal{G}(\tau)$, since the diagram is indexed by a contractible partially ordered set: in fact, the geometric realization of this partially ordered set is homeomorphic to τ). Let $S_0 = \{\sigma \in S : \sigma \subseteq \tau\}$ and let $S_1 = \{\sigma \in S : i(\sigma) = \tau\}$. The map θ is determined by a natural transformation between diagrams $S_0 \to \mathcal{C}$, and this natural transformation is invertible when restricted to S_1. To prove that θ is invertible, it suffices to show that S_1 is cofinal in S_0. This is a special case of the following more general assertion (applied in the case $M = \tau$):

Lemma 9. *Let M be a piecewise linear n-manifold with boundary, equipped with a triangulation S. Let S_1 be the collection of simplices of S which are not contained in ∂M. Then the inclusion $S_1 \hookrightarrow S$ is cofinal.*

Remark 10. Lemma 9 can be regarded as an analogue of the assertion that a manifold with boundary is always homotopy equivalent to its interior.

To prove Lemma 9, we work by induction on n. Fix a simplex $\sigma \in S$; we wish to show that the set $V_1 = \{\sigma' \in S_1 : \sigma \subseteq \sigma'\}$ has weakly contractible nerve. If $\sigma \in S_1$ this is obvious (since the subset above contains σ as a smallest element). Let us therefore assume that σ is a simplex of the boundary ∂M. Let $V = \{\sigma \in S : \sigma \subsetneq \sigma'\}$. Then V can be identified with the partially ordered set of simplices of $\mathrm{lk}(\sigma)$, which (since M is a PL manifold with boundary) is PL isomorphic to a disk D^m for $m < n$. We can identify V_0 with the subset of V consisting of simplices which are not contained in ∂D^m. Using the inductive hypothesis, we deduce that the inclusion $V_0 \to V$ is cofinal. Since V has weakly contractible nerve, so does V_0.

Proposition 8 implies that, for every $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, the canonical map

$$\Gamma(\mathcal{F}) \to \Gamma(i^* \mathcal{F})$$

is an equivalence. In particular, taking $\mathcal{F} = i_+ \mathcal{G}$, we obtain a canonical map

$$\Gamma(\mathcal{G}) \to \Gamma(i^* i_+ \mathcal{G}) \simeq \Gamma(i_+ \mathcal{G})$$

.

Proposition 11. *In the above situation, the map*

$$\Gamma(\mathcal{G}) \to \Gamma(i_+ \mathcal{G})$$

is an equivalence.

Proof. Let us assume for simplicity that $\mathcal{G}$ is compactly generated. Using Remark 5, we can assume that $\mathcal{F} = \mathcal{F}^{\sigma,C}$ for some simplex $\sigma \in S$. Then $i_+ \mathcal{F} \simeq \mathcal{F}^{\tau,C}$, where $\tau = i(\sigma)$. Choose a point $x \in X$ belonging to the interior of σ, so that x also belongs to the interior of τ. The calculation of Example 7 gives

$$\Gamma(\mathcal{G}) \simeq C^{(X, X - \{x\})} \simeq \Gamma(i_+ \mathcal{G}).$$

□

Definition 12. Let K be a polyhedron and $\mathcal{C}$ an ∞-category. We let $\mathrm{Shv}_{\mathrm{const}}(K; \mathcal{C})$ denote the direct limit $\varprojlim_T \mathrm{Shv}_T(X; \mathcal{C})$, where T ranges over all triangulations of X. We will refer to $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ as the *∞-category of constructible $\mathcal{C}$-valued sheaves on* K. If $\mathcal{C}$ is stable, we let $\mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C}) = \varinjlim_T \mathrm{Shv}^c_T(X; \mathcal{C})$; we will refer to the objects of $\mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})$ as *compactly supported* constructible $\mathcal{C}$-valued sheaves on X.