Introduction (Lecture 1)

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In this course, we will be concerned with variations on the following:

Question 1. Let X be a CW complex. When does there exist a homotopy equivalence $X \simeq M$, where M is a compact smooth manifold?

In other words, what is special about the homotopy type of a compact smooth manifold M? One special feature is obvious:

Fact 2. *Compact manifolds satisfy Poincare duality.*

Let us assume for simplicity that X is simply connected. If M is a compact smooth manifold homotopy equivalent to X, M is also simply connected and therefore orientable. A choice of orientation determines a fundamental homology class $[M] \in \mathrm{H}_n(M; \mathbf{Z})$, where n denotes the dimension of M. If $f : M \to X$ is a homotopy equivalence, then $[X] = f_*[X]$ is an element of $\mathrm{H}_n(X; \mathbf{Z})$ with the following property: for every integer q, the operation of cap product with $[X]$ induces an isomorphism

$$\mathrm{H}^q(X; \mathbf{Z}) \to \mathrm{H}_{n-q}(X; \mathbf{Z}).$$

This motivates the following definition:

Definition 3. Let X be a simply connected CW complex. We say that X is a *simply connected Poincare complex of dimension* n if there exists a homology class $[X] \in \mathrm{H}_n(X; \mathbf{Z})$ such that cap product with $[X]$ induces isomorphisms

$$\mathrm{H}^q(X; \mathbf{Z}) \to \mathrm{H}_{n-q}(X; \mathbf{Z})$$

for every integer q. In this case, we say that $[X]$ is a *fundamental class* of X.

Example 4. Any compact smooth manifold of dimension n is a Poincare complex of dimension n.

Remark 5. Taking $q = 0$ in Definition 3 and using our assumption that X is connected, we obtain an isomorphism

$$\mathbf{Z} \simeq \mathrm{H}^0(X; \mathbf{Z}) \to \mathrm{H}_n(X; \mathbf{Z})$$

given by $1 \mapsto [X]$. It follows that $\mathrm{H}_n(X; \mathbf{Z})$ must be a free abelian group of rank 1, and $[X]$ must be a generator of $\mathrm{H}_n(X; \mathbf{Z})$. Consequently, the fundamental class of X is well-defined up to sign.

Remark 6. If X is a simply connected Poincare complex of dimension n, then we have

$$\mathrm{H}_q(X; \mathbf{Z}) \simeq \mathrm{H}^{n-q}(X; \mathbf{Z}) \simeq 0$$

for $q > n$. In particular, n is uniquely determined by X: it is the largest degree of a nonvanishing homology group of X.

Remark 7. Using the fact that $\mathrm{H}^m(X; \mathbf{Z})$ vanishes for $m > n$ and that it is free when $m = n$, one can show that X is homotopy equivalent to a CW complex whose cells have dimension $\leq n$. However, we will not need this fact and we will not require that X itself have this property.

We can now give a partial answer to Question 1: if X is to be homotopy equivalent to a compact smooth manifold, then X must be a Poincare complex. We can therefore refine Question 1 (in the simply connected case) as follows:

Question 8. Let X be a simply connected Poincare complex of dimension n. When does there exist a homotopy equivalence $X \simeq M$, where M is a smooth manifold of dimension n?

To address Question 8, we make another observation: if M is a smooth manifold of dimension n, then M has a *tangent bundle* T_M, which is a real vector bundle of rank n over M. Moreover, the tangent bundle of M is closely connected with our discussion of Poincare duality.

We begin by considering the *normal bundle* of M. Choose an embedding $i : M \to \mathbb{R}^k$ for some large integer k. Let N_M denote the normal bundle to this embedding. By choosing a tubular neighborhood of M in $\mathbb{R}^k$, we can identify N_M with an open subset of $\mathbb{R}^k$. The Thom space $T(N_M)$ is given by the one-point compactification of N_M, given by $N_M \cup \{*\}$. We have a Thom-Pontryagin collapse map

$$c : S^k = \mathbb{R}^k \cup \{\infty\} \to T(N_M),$$

given by $c(v) = \begin{cases} v & \text{if } v \in N \\ * & \text{otherwise.} \end{cases}$ which determines an element $[c] \in \pi_k(T(N_M), \infty)$. Since M is simply connected, the normal bundle N is oriented. Choosing an orientation, we obtain a Thom isomorphism $\mathrm{H}_k(T(N_M), *; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z})$. Composing with the Hurewicz map, we obtain a homomorphism

$$\pi_k(T(N_M), *) \to \mathrm{H}_k(T(N_M), *; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z}).$$

The image of $[c]$ under this composite map is a fundamental homology class $[M]$. (with respect to the orientation determined by the choice of orientation on N_M).

The vector bundle N_M is not unique: it depends on a choice of embedding $M \hookrightarrow \mathbb{R}^k$. However, the *spectrum* $\Sigma^{\infty - k} T(N_M)$ is uniquely determined. We have a canonical exact sequence of vector bundles

$$T_M \to T_{\mathbb{R}^n}|M \to N_M.$$

Choosing a splitting of this exact sequence, we obtain a direct sum decomposition $N_M \oplus T_M \simeq \underline{R}^k$, where $\underline{R}^k$ denotes the trivial vector bundle of rank k. It follows that the spectrum $\Sigma^{\infty - k} T(N_M)$ can be identified with the *Thom spectrum* M^{-T_M} of the virtual vector bundle $-T_M$ on M. The element $[c] \in \pi_k(T(N_M), \infty)$ determines an element $\eta_M \in \pi_0 M^{-T_M}$, which is independent of the choice of embedding i.

We can summarize the above discussion as follows:

Fact 9. *Let M be a simply connected smooth manifold of dimension n. Then there exists a vector bundle ζ on M of dimension n (namely, the tangent bundle T_M) and a class $\eta_M \in \pi_0 M^{-\zeta}$ such that the image of η_M under the composite map*

$$\pi_0(M^{-\zeta}) \to \mathrm{H}_0(M^{-\zeta}; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z})$$

is a fundamental class of M. Here the second map is the Thom isomorphism (determined by a choice of orientation of ζ).

This gives us another necessary condition that a simply connected CW complex X must satisfy if X is to be homotopy equivalent to a manifold of dimension n: there must exist a vector bundle ζ on X and a homotopy class $\eta_X \in \pi_0 X^{-\zeta}$ whose image $[X] \in \mathrm{H}_n(X; \mathbf{Z})$ exhibits X as a Poincare complex of dimension n. We may therefore refine our question yet again:

Question 10. Let X be a simply connected Poincare complex of dimension n. Suppose we are given a vector bundle ζ of dimension n on X and a homotopy class $\eta_X \in \pi_0 X^{-\zeta}$ whose image in $\mathrm{H}_n(X; \mathbf{Z})$ is a fundamental homology class for X. Does there exist a smooth manifold M of dimension n and a homotopy equivalence $f : M \to X$ such that $f^* \zeta$ is (stably) isomorphic to T_M and $f^* \eta_X = \eta_M \in \pi_0 M^{-T_M}$?

We now give the answer to Question 10 in the simplest case. Assume that X is a simply connected Poincare complex of dimension $n = 4k$. In this case, we have a symmetric bilinear form

$$\langle , \rangle : \mathrm{H}^{2k}(X;\mathbb{R}) \times \mathrm{H}^{2k}(X;\mathbb{R}) \rightarrow \mathrm{H}^{4k}(X;\mathbb{R}) \stackrel{[X]}{\rightarrow} \mathbb{R}$$

and Poincare duality ensures that this form is nondegenerate. We may therefore choose an orthogonal basis $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ for $\mathrm{H}^{2k}(X;\mathbb{R})$ satisfying $\langle x_i, x_i \rangle = 1$ and $\langle y_i, y_i \rangle = -1$. The difference $a - b$ is called the *signature* of X, and will be denoted by σ_X. Note that the sign of σ_X depends on a choice of fundamental class for X.

If M is a compact smooth manifold of dimension $n = 4k$, then the signature of M is given by the *Hirzebruch signature formula*. Namely, there is a formula

$$\sigma_M = L(p_1(T_M), p_2(T_M), \ldots, p_k(T_M))[M].$$

Here $L(p_1(T_M), p_2(T_M), \ldots, p_k(T_M))$ denotes some polynomial in the Pontryagin classes $p_i(T_M)$ (note that the right hand side of this formula also depends up to sign on our choice of orientation of M). For example, when $n = 4$ we have $\sigma_M = \frac{p_1(T_M)}{3}[M]$, and when $n = 8$ we have

$$\sigma_M = \frac{7p_2(T_M) - p_1(T_M)^2}{45}[M].$$

Remark 11. If we choose a connection on the manifold M, then we can use Chern-Weil theory to obtain explicit differential forms representing the Pontryagin classes of the tangent bundle T_M. Consequently, the signature of M can be computed by integrating over M an explicitly given n-form on M. We can therefore regard the Hirzebruch signature formula as saying that there is a *purely local* formula for the signature, which is defined *a priori* as a global invariant of M.

Remark 12. Here is a very rough heuristic justification for why there should exist a Hirzebruch signature formula. If X is a Poincare complex of dimension $4k$, then the signature σ_X is defined because we can define an intersection form using Poincare duality. If X is a manifold, the Poincare duality is satisfied for a "local" reason, so we might expect to obtain a "local" formula for σ_X. Later in this course, we will prove the Hirzebruch signature formula by making this heuristic more precise.

This gives us one further condition that a triple, $(X, \zeta, \nu_X \in \pi_0 X^{-\zeta})$ must satisfy to obtain an affirmative answer to Question 10. Namely, we must have

$$\sigma_X = L(p_1(\zeta), p_2(\zeta), \ldots, p_k(\zeta))[X].$$

Simply-connected surgery provides a converse in high dimensions:

Theorem 13 (Browder, Novikov?). *Let X be a simply connected Poincare complex of dimension $4k > 4$, let ζ be a vector bundle (of rank $4k$) on X, and let $\eta_X \in \pi_0 X^{-\zeta}$ be such that the image of η_X in $\mathrm{H}_{4k}(X;\mathbf{Z})$ is a fundamental class. Then Question 10 has an affirmative answer if and only if $\sigma_X = L(p_1(\zeta), p_2(\zeta), \ldots, p_k(\zeta))[X]$: that is, if and only if X satisfies the Hirzebruch signature formula.*

Theorem 13 is a prototype for the type of result we would like to obtain in this class. We will pursue a number of variations:

(*a*) We can contemplate Question 1 for Poincare complexes X which are not assumed to be simply connected.

(*b*) Question 1 concerns the *existence* of a manifold M in the homotopy type of X. If the answer is affirmative, one can further ask if M is unique.

Let us briefly describe how problem (b) can be attacked. Suppose that we are given a Poincare complex X and a pair of homotopy equivalences $f : X \to M$, $g : X \to M'$, where M and M' are compact manifolds of dimension n. We can then consider the "double mapping cylinder" $Y = M \coprod_{X \times \{0\}} (X \times [0,1]) \coprod_{X \times \{1\}} M'$. The pair $(Y, M \amalg M')$ satisfies a relative version of Poincare duality. This suggests that we might look for an $(n+1)$-manifold B with boundary $M \amalg M'$ and a homotopy equivalence $(B, M \amalg M') \to (Y, M \amalg M')$ which restricts to the identity map on M and M'. If we can solve this problem, then B is an *h-cobordism* from M to M': that is, a bordism from M to M' such that the inclusions $M \hookrightarrow B \hookleftarrow M'$ are homotopy equivalences. If $n \geq 5$ and M is simply connected, then the h-cobordism theorem guarantees the existence of a diffeomorphism $B \simeq M \times [0,1]$, which in particular gives a diffeomorphism $M \simeq M'$.

In summary, the problem of deciding whether M is unique can be regarded as another of roughly the same type as Question 1. This motivates considering two more types of variations of Question 1:

(c) Rather than considering the *absolute* case of a Poincare complex X, we should consider the problem of proving that a pair of spaces $(X, \partial X)$ is homotopy equivalent to a manifold with boundary.

(d) We should not restrict our attention to the case of a fixed dimension n: a lot of information about the classification of manifolds of dimension n can be obtained by thinking about manifolds of dimension $> n$. In particular, we should not restrict our attention to the case where n is a multiple of four. (However, we will retain the assumption that $n > 4$: this is the domain of *high-dimensional topology* where techniques of surgery work well).

If M is not simply connected, then an h-cobordism from M to M' does not generally guarantee that M and M' are diffeomorphic: one encounters an algebraic obstruction called the *Whitehead torsion*. This is an interesting story, but not one we will discuss in this class: we will be content to give the classification of manifolds in a homotopy type *up to h-cobordism*.

In fact, we will do more. Suppose that M and M' are as above, and that Y has the homotopy type of an h-cobordism B from M to M'. We might then ask a higher-order uniqueness question: to what extent is the bordism B uniquely determined? To ask these questions in an organized way, it is convenient to introduce the *structure space* $S(X)$ of a Poincare complex X. This is a space whose connected components are given by manifolds M with a homotopy equivalence $M \to X$, up to h-cobordism. Question 1 is the question of whether or not $S(X)$ is nonempty, and the uniqueness problem amounts to the question of whether or not $S(X)$ is connected. Better still, we might try to discuss the entire homotopy type of $S(X)$.

(e) We can ask an analogue of Question 1 for manifolds equipped with various structure. Suppose, for example, that we wanted to find a spin manifold in the homotopy type of the Poincare complex X. The collection of h-cobordism classes of such manifolds can be described as connected components of a slightly different structure space $S_{\mathrm{Spin}}(X)$. By forgetting spin structures, we obtain a map of structure spaces $\theta : S_{\mathrm{Spin}}(X) \to S(X)$. Giving a spin structure on a manifold M is equivalent to giving a spin structure on its tangent bundle T_M: that is, to reducing the structure group of M from the orthogonal group $\emptyset(n)$ to the spin group $\mathrm{Spin}(N)$. Consequently, the homotopy fibers of the map θ are easy to describe. Consequently, the problem of determining the homotopy type of $S_{\mathrm{Spin}}(X)$ can be reduced to the problem of determining the homotopy type of $S(X)$.

What we have denoted by $S(X)$ should really be denoted $S_{\mathrm{Sm}}(X)$, the *smooth* structure space, because in the above discussion we required all manifolds to be smooth. We can also define a *topological* structure space $S_{\mathrm{Top}}(X)$ by considering topological manifolds with a homotopy equivalence to X. By forgetting smooth structures, we obtain a map of structures spaces $S_{\mathrm{Sm}}(X) \to S_{\mathrm{Top}}(X)$. The relationship between $S_{\mathrm{Sm}}(X)$ and $S_{\mathrm{Top}}(X)$ is similar to the relationship between $S_{\mathrm{Spin}}(X)$ and $S_{\mathrm{Sm}}(X)$: according to smoothing theory, for topological manifolds M of dimension > 4, giving a smooth structure on M is equivalent giving a vector bundle structure on the topological tangent bundle T_M. In other words, to classify smooth manifolds in the homotopy type of X we can proceed by first classifying the topological manifolds in the homotopy type of X and then studying the problem of smoothing them, where the second step reduces to a purely homotopy-theoretic problem. Put another way, there is a good homotopy-theoretic understanding of the homotopy fibers of the map $S_{\mathrm{Sm}}(X) \to S_{\mathrm{Top}}(X)$.

However, there is a much more compelling reason to work with topological manifolds rather than smooth manifolds: the topological versions of these questions often have nicer answers. For example, there is only one topological manifold in the homotopy type of the n-sphere S^n (by the generalized Poincare conjecture), but this topological manifold admits many different smooth structures (exotic spheres). The ultimate algebraic description of structure spaces which we obtain will be cleanest in the topological category. For example, $S_{\mathrm{Sm}}(X)$ is just a space, but we will later see that $S_{\mathrm{Top}}(X)$ is an infinite loop space (if nonempty). A concrete consequence of this is that if we fix a topological manifold M, then the collection of h-cobordism classes of manifolds in with a homotopy equivalence to M has the structure of an abelian group.

Our ultimate goal in this course is to obtain a purely homotopy theoretic description of the structure space $S(X)$ of a Poincare complex X. Though we are not yet ready to formulate this description precisely, let us assert that it has the same basic form as the statement of Theorem 13. Namely, we will associate to X a certain invariant σ, called the *visible symmetric signature of* X. We will then show that finding a manifold in the homotopy type of X amounts to verifying a "local formula" for this invariant, generalizing the Hirzebruch signature formula (see Remark 11). The fine print is that this invariant is not an integer, but something more sophisticated. Explaining exactly what that "something" is will require us to develop the apparatus of *algebraic L-theory*. That is our objective for the first half of this course. In the second half, we will return to the theory of manifolds, using the algebraic apparatus to prove a very general version of Theorem 13.