Koszul Duality in Algebraic Geometry (Lecture 24)

April 7, 2014

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Let us assume for simplicity that G is everywhere reductive. Associated to G, we have prestack morphisms

$$\mathrm{Ran}_G(X) \xrightarrow{\phi} \mathrm{Ran}(X)$$

$$\mathrm{Ran}^G(X) \xrightarrow{\psi} \mathrm{Ran}(X).$$

Recall that the objects of $\mathrm{Ran}_G(X)$ are tuples $(R, S, \mu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, S is a nonempty finite set, $\mu : S \to X(R)$ is a map, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ on $X - |\mu|$. The objects of $\mathrm{Ran}^G(X)$ are tuples $(R, T, \nu, \mathcal{P})$ where R is a finitely generated k-algebra, T is a nonempty finite set, $\nu : T \to X(R)$ is a map, and $\mathcal{P}$ on a G-bundle on $X - |\nu|$. These two maps have different variance properties. Given a nonempty finite set S, the projection map

$$\mathrm{Ran}_G(X)_S = \mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} X^S \to X^S$$

is Ind-proper (since G is everywhere reductive); for a nonempty finite set T, the map

$$\mathrm{Ran}^G(X)^T = \mathrm{Ran}^G(X) \times_{\mathrm{Ran}(X)} X^T \to X^T$$

is instead a smooth morphism of algebraic stacks. Given a surjection of nonempty finite sets $S \to S'$, we have a natural map

$$\mathrm{Ran}_G(X)_{S'} \to X^{S'} \times_{X^S} \mathrm{Ran}_G(X)_S,$$

while a surjection $T \to T'$ instead induces a map

$$\mathrm{Ran}^G(X)^{T'} \leftarrow X^{T'} \times_{X^T} \mathrm{Ran}^G(X)^T.$$

As a consequence of these differences, the maps ϕ and ψ can be used to produce two different types of sheaves on $\mathrm{Ran}(X)$. We have previously defined the !-sheaf $\mathcal{B} = [\mathrm{Ran}^G(X)]_{\mathrm{Ran}(X)}$, which we can think of informally as given by $\psi_* \psi^* \omega_{\mathrm{Ran}(X)}$. Similarly, one can construct a $*$-sheaf $\mathcal{A} = \phi_* \phi^* \underline{\mathbf{Z}_\ell}_{\mathrm{Ran}(X)}$. Our goal in this lecture is to describe how these constructions are related. As we have hinted earlier, these objects are related by a *covariant* form a Verdier duality, at least after an appropriate normalization.

We begin with an informal discussion. Consider the prestack $\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$. Let us informally identify the points of $\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ with pairs (S, T), where S and T are nonempty finite subsets of X. Given such a pair, any G-bundle $\mathcal{P}$ on X can be restricted to a G-bundle on T. This construction determines a commutative diagram

$$\begin{array}{ccc} \mathrm{Ran}_G(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) & \longrightarrow & \mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}^G(X) \\ & \searrow \quad \swarrow & \\ & \mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X). & \end{array}$$

Ignoring the distinction between !-sheaves and $*$-sheaves for the moment, we can think of this diagram as supplying a map

$$\theta : \underline{\mathbf{Z}_{\ell}}_{\mathrm{Ran}(X)} \boxtimes \mathcal{B} \to \mathcal{A} \boxtimes \omega_{\mathrm{Ran}(X)}.$$

For every pair (S, T), we can pass to the stalk at S and costalk at T to obtain a map

$$\theta_{S,T} = \bigotimes_{t \in T} C^*(\mathrm{BG}_t; \mathbf{Z}_\ell) \to \bigotimes_{s \in S} C^*(\mathrm{Gr}_s^G; \mathbf{Z}_\ell).$$

Geometrically, this map arises from a map of prestacks

$$\rho_{S,T} : \prod_{s \in S} \mathrm{Gr}_s^G \to \prod_{t \in T} \mathrm{BG}_t .$$

Note that if $S \neq T$, then this map exhibits some degenerate behavior. For example, if there exists an element $s_0 \in S$ which does not belong to T, then the map $\rho_{S,T}$ factors through the product $\prod_{s \neq s_0} \mathrm{Gr}_s^G$, which we can think of as parametrizing G-bundles on $X - \{s_0\}$ with a trivialization on $X - S$ (in order to restrict a G-bundle to the set T, we do not need it to be defined at the point s_0). Similarly, if there exists an element $t_0 \in T$ which does not belong to S, then the map $\rho_{S,T}$ factors through $\prod_{t \neq t_0} \mathrm{BG}_t$ (since any G-bundle trivial on $X - S$ will be trivial at the point t_0). In either case, we conclude that the composite map

$$\begin{aligned} \bigotimes_{t \in T} C^*_{\mathrm{red}}(\mathrm{BG}_t; \mathbf{Z}_\ell) &\to \bigotimes_{t \in T} C^*(\mathrm{BG}_t; \mathbf{Z}_\ell) \\ &\xrightarrow{\theta_{S,T}} \bigotimes_{s \in S} C^*(\mathrm{Gr}_s^G; \mathbf{Z}_\ell) \\ &\to \bigotimes_{s \in S} C^*_{\mathrm{red}}(\mathrm{Gr}_s^G; \mathbf{Z}_\ell) \end{aligned}$$

vanishes.

It is possible to introduce "reduced versions" of the sheaves $\mathcal{A}$ and $\mathcal{B}$, which we will denote by $\mathcal{A}_{\mathrm{red}}$ and $\mathcal{B}_{\mathrm{red}}$, whose (co)stalks are given by

$$S^* \mathcal{A}_{\mathrm{red}} = \bigotimes_{s \in S} C^*(\mathrm{Gr}_s^G; \mathbf{Z}_\ell) \qquad T^! \mathcal{B}_{\mathrm{red}} = \bigotimes_{t \in T} C^*(\mathrm{BG}_t; \mathbf{Z}_\ell).$$

An elaboration of the above argument shows that θ induces a map

$$\theta_{\mathrm{red}} : \underline{\mathbf{Z}_{\ell}}_{\mathrm{Ran}(X)} \boxtimes \mathcal{B}_{\mathrm{red}} \to \mathcal{A}_{\mathrm{red}} \boxtimes \omega_{\mathrm{Ran}(X)}$$

which vanishes away from the diagonal of $\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$. Heuristically, this means that θ_{red} factors through a map

$$\underline{\mathbf{Z}_{\ell}}_{\mathrm{Ran}(X)} \boxtimes \mathcal{B}_{\mathrm{red}} \to \Delta_* \Delta^! \mathcal{A}_{\mathrm{red}} \boxtimes \omega_{\mathrm{Ran}(X)},$$

which we can identify with a map

$$\mathcal{B}_{\mathrm{red}} = \Delta^*(\underline{\mathbf{Z}_{\ell}}_{\mathrm{Ran}(X)} \boxtimes \mathcal{B}_{\mathrm{red}}) \to \Delta^!(\mathcal{A}_{\mathrm{red}} \boxtimes \omega_{\mathrm{Ran}(X)}) \simeq \mathcal{A}_{\mathrm{red}} .$$

The main idea of our proof is to show that this map is an equivalence. However, this does not quite make sense as we have formulated it: the right hand side is a !-sheaf on $\mathrm{Ran}(X)$, and the left hand side is a $*$-sheaf on $\mathrm{Ran}(X)$. Moreover, many of the objects which appeared in the above discussion (like the external tensor product $\mathcal{A}_{\mathrm{red}} \boxtimes \omega_{\mathrm{Ran}(X)}$) need to be interpreted as some sort of hybrid between $*$-sheaves and !-sheaves. It will therefore be convenient to recast the above discussion in a less symmetrical way (essentially by "pushing forward" all of our sheaves onto the second copy of $\mathrm{Ran}(X)$), which involves only !-sheaves on $\mathrm{Ran}(X)$.

Let us now dispense with heuristics and describe the strategy we will actually pursue. For every nonempty finite set S, we let $\mathrm{Ran}_G(X)_S$ denote the fiber product $\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} X^S$. We then have maps of Ran-prestacks

$$\mathrm{Ran}_G(X)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \to X^S \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \to \mathrm{Ran}^G(X),$$

depending functorially on S. We therefore obtain maps of !-sheaves

$$\mathcal{B} \to C^*(X^S; \mathbf{Z}_\ell) \otimes C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \to C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell),$$

depending functorially on S. Passing to chiral homology, we obtain maps

$$\int_{\mathrm{Ran}(X)} \mathcal{B} \stackrel{\alpha_S}{\to} C^*(X^S; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \stackrel{\beta_S}{\to} C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell).$$

The inverse limit of the maps β_S as S varies can be identified with the natural map $C^*(\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell)$: this is predual to the equivalence

$$\begin{aligned} C_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) &\simeq C_*(\mathrm{Ran}(X); \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\ &\simeq C_*(\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\ &\to C_*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell), \end{aligned}$$

supplied by nonabelian Poincare duality. The inverse limit of the maps α_S as S varies can be identified with the map

$$\int_{\mathrm{Ran}(X)} \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

that we discussed in the previous lecture. Consequently, we are reduced to proving the following:

Proposition 1. *The induced map*

$$\int_{\mathrm{Ran}(X)} \mathcal{B} \to \varprojlim_S C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

We will prove this by factoring the the composite map

$$\xi : \mathrm{Ran}_G(X)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \to X^S \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \to \mathrm{Ran}^G(X)$$

in a different way. For this, we need an auxiliary construction.

Construction 2. Fix a nonempty finite set S and a pair of subsets $K_- \subseteq K_+ \subseteq S$. We define a prestack $\mathcal{C}(K_-, K_+)$ as follows:

- The objects of $\mathcal{C}(K_-, K_+)$ are tuples $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, T is a nonempty finite set, $\mu : S \to X(R)$ and and $\nu : T \to X(R)$ are maps of sets such that $|\mu(K_+)| \cap |\nu(T)| = \emptyset$, $\mathcal{P}$ is a G-bundle on $X_R - |\mu(S_-)|$ which can be extended to a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over $X_R - |\mu(S)|$.
- A morphism from $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ to $(R', \mu', \nu' : T' \to X(R'), \mathcal{P}', \gamma')$ consists of a k-algebra homomorphisms $\phi : R \to R'$ such that μ' is given by the composition $S \stackrel{\mu}{\to} X(R) \stackrel{X(\phi)}{\to} X(R')$, a surjection of finite sets $\lambda : T \to T'$ which fits into a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\lambda} & T' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R'), \end{array}$$

and a G-bundle isomorphisms between $\mathcal{P} \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ and $\mathcal{P}'$ over the scheme $X_{R'} - |\mu'(K_-)|$ which carries γ to γ'.

Remark 3. If the set S and the subset $K_+ \subseteq S$ are fixed, then we can regard $\mathcal{C}(K_-, K_+)$ as a covariant functor of K_-: for every inclusion $K_- \subseteq K'_- \subseteq K_+$, we have a forgetful functor

$$\mathcal{C}(K_-, K_+) \to \mathcal{C}(K'_-, K_+)$$

given by restriction of G-bundles. Here it is helpful to think of $\mathcal{C}(K'_-, K_+)$ as the quotient of $\mathcal{C}(K_-, K_+)$ obtained by identifying G-bundles which differ away from the image of K'_+ in X.

Remark 4. If the set S and the subset $K_- \subseteq S$ are fixed, then we can regard $\mathcal{C}(K_-, K_+)$ as a contravariant functor of K_+: for every inclusion $K_- \subseteq K_+ \subseteq K'_+$, we can identify $\mathcal{C}(K_-, K'_+)$ with a full subcategory of $\mathcal{C}(K_-, K_+)$ (given by those objects $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ which satisfy the additional condition that $\mu(K'_+) \cap \nu(T) = \emptyset$).

Example 5. If $K_- = K_+ = \emptyset$, then we can identify $\mathcal{C}(K_-, K_+)$ with the product $\operatorname{Ran}_G(X)_S \times_{\operatorname{Spec} k} \operatorname{Ran}(X)$.

Definition 6. Fix a nonempty finite set S. We let $\operatorname{Ran}^{\dagger}_G(X)_S$ denote the category obtained via the Grothendieck construction on the functor $(K_-, K_+) \mapsto \mathcal{C}(K_-, K_+)$. More precisely, we have the following:

- The objects of $\operatorname{Ran}^{\dagger}_G(X)_S$ are tuples $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, K_- and K_+ are subsets of S with $K_- \subseteq K_+$, T is a nonempty finite set, $\mu : S \to X(R)$ and and $\nu : T \to X(R)$ are maps of sets such that $|\mu(K_+)| \cap |\nu(T)| = \emptyset$, $\mathcal{P}$ is a G-bundle on $X_R - |\mu(K_-)|$ which can be extended to a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over $X_R - |\mu(S)|$.

- There are no morphisms

$$(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma) \to (R', K'_-, K'_+, \mu', \nu' : T' \to X(R'), \mathcal{P}', \gamma')$$

  unless $K'_- \subseteq K_- \subseteq K_+ \subseteq K'_+$. If this condition is satisfied, then a morphism from $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ to $(R', K'_-, K'_+, \mu', \nu' : T' \to X(R), \mathcal{P}', \gamma')$ consists of a k-algebra homomorphism $\phi : R \to R'$ carrying μ to μ', a surjection of finite sets $\lambda : T \to T'$ which fits into a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\lambda} & T' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R'), \end{array}$$

  and a G-bundle isomorphism between $\mathcal{P}$ and $\mathcal{P}'$ over the scheme $X_{R'} - |\mu'(K'_-)|$ which carries γ to γ'.

The construction $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma) \mapsto (R, T, \nu, \mathcal{P}\,|_{|\nu(T)|})$ determines a forgetful functor $f_S : \operatorname{Ran}^{\dagger}_G(X)_S \to \operatorname{Ran}^G(X)$. We let $\mathcal{B}_S$ denote the lax !-sheaf on $\operatorname{Ran}(X)$ given by the formula

$$\mathcal{B}^{(T)}_S = [\operatorname{Ran}^{\dagger}_G(X)_S \times_{\operatorname{Ran}(X)} X^T]_{X^T}.$$

Note that the map f_S induces a map of lax !-sheaves $\mathcal{B} \to \mathcal{B}_S$, depending functorially on S. Moreover, the identification $\mathcal{C}(\emptyset, \emptyset) \simeq \operatorname{Ran}_G(X)_S \times_{\operatorname{Spec} k} \operatorname{Ran}(X)$ determines a fully faithful embedding

$$\operatorname{Ran}_G(X)_S \times_{\operatorname{Spec} k} \operatorname{Ran}(X) \hookrightarrow \operatorname{Ran}^{\dagger}_G(X)_S,$$

which induces a pullback map $\mathcal{B}_S \to C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Ran}_G(X)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) & \longrightarrow & \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ran}^{\dagger}_G(X)_S & \longrightarrow & \mathrm{Ran}^G(X), \end{array}$$

we see that the map ξ of Proposition 1 can be identified with the composition

$$\begin{aligned} \int_{\mathrm{Ran}(X)} \mathcal{B} & \xrightarrow{\xi'} \int_{\mathrm{Ran}(X)} \varprojlim_S \mathcal{B}_S \\ & \xrightarrow{\xi''} \varprojlim_S \int_{\mathrm{Ran}(X)} (C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}) \\ & \simeq \varprojlim_S C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell). \end{aligned}$$

We are therefore reduced to proving the following pair of assertions:

Proposition 7. *The map ξ'' is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.*

Proposition 8. *The canonical map $\mathcal{B} \to \varprojlim_S \mathcal{B}_S$ is an equivalence of $!$-sheaves on $\mathrm{Ran}(X)$.*

The proof of Proposition 7 is mostly formal: the difficulty lies in showing that passage to the inverse limit over S "commutes" with passage to chiral homology. In terms of our heuristic picture, this is because the $*$-sheaf $\mathcal{A}_{\mathrm{red}}$ is generated by compactly supported sections: in fact, in any given degree, the cohomologies of the sheaf $\mathcal{A}_{\mathrm{red}}$ are supported on the substack $\mathrm{Ran}(X)_{\leq n}$ for $n \gg 0$. We will not present the details in class.

Proposition 8 can be regarded as a local calculation on the Ran space, which relates the cohomology of the Grassmannians $\mathrm{Gr}_{G,x}$ to the cohomology of the classifying stacks BG_y. We will return to this in the next lecture.