The Product Formula (Lecture 19)

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Throughout this lecture, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, and a smooth affine group scheme G over X.

Construction 1. We define a category $\mathrm{Ran}^G(X)$ as follows:

- The objects of $\mathrm{Ran}^G(X)$ are quadruples $(R, T, \nu, \mathcal{P})$ where R is a finitely generated k-algebra, T is a nonempty finite set, $\nu : T \to X(R)$ is a map of sets, and $\mathcal{P}$ is a G-bundle on the divisor $|\nu(T)| \subseteq X_R$ determined by ν.
- A morphism from $(R, T, \nu, \mathcal{P})$ to $(R', T', \nu', \mathcal{P}')$ in the category $\mathrm{Ran}^G(X)$ consists of a morphism $(R, T, \nu) \to (R', T', \nu')$ in $\mathrm{Ran}(X)$, together with a G-bundle isomorphism of $\mathcal{P}'$ with $|\nu'(T')| \times_{|\nu(T)|} \mathcal{P}$.

We will regard $\mathrm{Ran}^G(X)$ as a prestack via the forgetful functor $(R, T, \nu, \mathcal{P}) \mapsto R$. Note that there is an evident forgetful functor $\mathrm{Ran}^G(X) \to \mathrm{Ran}(X)$. For each nonempty finite set T, we let $\mathrm{Ran}^G(X)^{(T)}$ denote the fiber product $\mathrm{Ran}^G(X) \times_{\mathrm{Ran}(X)} X^T$.

Example 2. The prestack $\mathrm{Ran}^G(X)^{(1)}$ can be identified with the classifying stack BG: an R-valued point of $\mathrm{Ran}^G(X)^{(1)}$ is given by an R-valued point of X together with a G-bundle on $\operatorname{Spec} R$.

Each of the prestacks $\mathrm{Ran}^G(X)^{(1)}$ is actually an Artin stack, which can be identified with the classifying stack of a smooth affine group scheme over X^T (given by the Weil restriction of G along an "incidence correspondence" between X and X^T).

Remark 3. Let $\alpha : T \to T'$ be a surjection of finite sets. Then α induces a diagonal map $\delta_{T/T'} : X^{T'} \to X^T$. For any map $\nu' : T' \to X(R)$, we have $|\nu'(T')| \subseteq |(\nu' \circ \alpha)(T)| \subseteq X_R$. Consequently, any G-bundle on $|(\nu' \circ \alpha)(T)|$ determines a G-bundle on $|\nu'(T')|$. This observation determines a map of prestacks

$$X^{T'} \times_{X^T} \mathrm{Ran}^G(X)^T \to \mathrm{Ran}^G(X)^{T'}.$$

Construction 4. For each nonempty finite set T, we let $\mathcal{B}^{(T)}$ denote the ℓ-adic sheaf on X^T given by the formula

$$\mathcal{B}^{(T)} = [\mathrm{Ran}^G(X)^T]_{X^T}.$$

Note that if $\alpha : T \to T'$ is a surjection of nonempty finite sets, then Remark 3 determines a map of ℓ-adic sheaves

$$\begin{aligned}
\mathcal{B}^{(T')} &= [\mathrm{Ran}^G(X)^{T'}]_{X^{T'}} \\
&\to [X^{T'} \times_{X^T} \mathrm{Ran}^G(X)^T]_{X^T} \\
&\to \delta^!_{T/T'} [\mathrm{Ran}^G(X)^T]_{X^T} \\
&= \delta^!_{T/T'} \mathcal{B}^{(T)}
\end{aligned}$$

These maps exhibit $\{\mathcal{B}^{(T)}\}_{T \in \mathrm{Fin}^s}$ as a lax !-sheaf on $\mathrm{Ran}(X)$, in the sense of the previous lecture. We will denote this lax !-sheaf by $\mathcal{B}$.

Remark 5. We will see later that $\mathcal{B}$ is actually a !-sheaf on $\mathrm{Ran}(X)$.

Remark 6. Every G-bundle on X determines a G-bundle on every divisor $D \subseteq X$. This observation determines a map of prestacks

$$\mathrm{Ran}(X) \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X) \to \mathrm{Ran}^G(X).$$

In particular, for every nonempty finite set T, we obtain a map

$$X^T \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X) \to \mathrm{Ran}^G(X)^{(T)}$$

of prestacks over X^T, which induces a map

$$\begin{aligned} \mathcal{B}^{(T)} &= [\mathrm{Ran}^G(X)^{(T)}]_{X^T} \\ &\to [X^T \times_{\mathrm{Spec}\, k} \mathrm{Bun}_G(X)]_{X^T} \\ &\simeq C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{X^T} \end{aligned}$$

of ℓ-adic sheaves on X^T. This construction depends functorially on T, and determines a map $\mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$ in the ∞-category $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ of the previous lecture.

Remark 7. For any object $M \in \mathrm{Mod}_{\mathbf{Z}_\ell}$, we have

$$\begin{aligned} \int M \otimes \omega_{\mathrm{Ran}(X)} &\simeq M \otimes_{\mathbf{Z}_\ell} \int \omega_{\mathrm{Ran}(X)} \\ &\simeq M \otimes_{\mathbf{Z}_\ell} \varinjlim_{T \in \mathrm{Fin}^s} C^*(X^T; \omega_{X^T}) \\ &\simeq M \otimes_{\mathbf{Z}_\ell} \varinjlim_{T \in \mathrm{Fin}^s} C_*(X^T; \mathbf{Z}_\ell) \\ &\simeq M \otimes_{\mathbf{Z}_\ell} C_*(\mathrm{Ran}(X); \mathbf{Z}_\ell) \\ &\simeq M. \end{aligned}$$

Consequently, we can view Remark 6 as defining a map

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

We can now state the second main theorem of this course:

Theorem 8 (Product Formula)**.** *Suppose that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then the map*

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is a quasi-isomorphism.

Remark 9. Let η be a k-valued point of $\mathrm{Ran}(X)$, corresponding to a finite set T and a map $\nu : T \to X(k)$. Then we can define the costalk $\eta^! \mathcal{B}$ by the formula

$$\eta^! \mathcal{B} = \nu^! \mathcal{B}^{(T)},$$

where we abuse notation by identifying ν with a map $\mathrm{Spec}\, k \to X^T$.

Assume for simplicity that ν is injective. Then we have

$$\begin{aligned}
\eta^! \mathcal{B} &= \nu^! \mathcal{B}^{(T)} \\
&= \nu^! [\mathrm{Ran}^G(X)^{(T)}]_{X^T} \\
&\simeq C^*(\mathrm{Ran}^G(X)^{(T)} \times_{X^T} \mathrm{Spec}\, k; \mathbf{Z}_\ell) \\
&\simeq C^*(\prod_{t \in T} \mathrm{BG}_{\nu(t)}; \mathbf{Z}_\ell) \\
&\simeq \prod_{t \in T} C^*(\mathrm{BG}_{\nu(t)}; \mathbf{Z}_\ell).
\end{aligned}$$

Note that we have a canonical map

$$\eta^! \mathcal{B} = \nu^! \mathcal{B}^{(T)} \to C^*(X^T; \mathcal{B}^{(T)}) \to \int \mathcal{B}\,.$$

Heuristically, we can think of $\int \mathcal{B}$ as a "continuous colimit" of the costalks $\eta^! \mathcal{B}$, where η ranges over $\mathrm{Ran}(X)$. In other words, we can think of $\int \mathcal{B}$ as a "continuous tensor product"

$$\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell).$$

From this point of view, Theorem 8 is a kind of "continuous Künneth formula", which reflects the idea that $\mathrm{Bun}_G(X)$ can be described heuristically as a product $\prod_{x \in X} \mathrm{BG}_x$.