Course Syllabus for Math 281: Algebraic K-theory and Manifold Topology

Course Description

Let M and N be smooth closed manifolds of dimension n. An *h-cobordism* from M to N is a compact smooth manifold B of dimension $(n+1)$ with boundary $\partial B \simeq M \amalg N$ having the property that the inclusion maps $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences. If $n \geq 5$ and the manifold M is simply connected, then the celebrated *h-cobordism theorem* of Smale asserts that B is diffeomorphic to a product $M \times [0,1]$ (and, in particular, M is diffeomorphic to N).

If M is not simply connected, then it is generally not true that any h-cobordism B from M to N is diffeomorphic to a product $M \times [0,1]$. In fact, one can introduce an algebraic invariant $\tau(B)$ (called the *Whitehead torsion* of B) belonging to an abelian group $Wh(M)$ (called the *Whitehead group* of M). The invariant $\tau(B)$ vanishes whenever B is diffeomorphic to a product $M \times [0,1]$, and the converse holds provided that the dimension of M is greater than 5 (this statement is known as the *s-cobordism theorem*). The Whitehead group $Wh(M)$ depends only on the fundamental group of M and vanishes when M is simply connected, so that the s-cobordism theorem implies the h-cobordism theorem.

For many purposes, it is useful to know whether there is an analogue of the s-cobordism theorem for *families* of manifolds. Fix a compact smooth n-dimensional manifold M, a finite cell complex X, and suppose we given a fiber bundle $B \to X$ where each of the fibers B_x is an h-cobordism from M to some other n-manifold N_x (where the smooth structures vary continuously with x). Under what circumstances can we deduce that B is equivalent to a product $M \times [0,1] \times X$ (so that the fiber bundle $\{N_x\}_{x \in X}$ is trivial)? In this course, we will study an analogue of the Whitehead torsion which is adapted to the parametrized setting:

- In place of the Whitehead group $W(M)$, we will consider an infinite loop space $\mathbf{Wh}(M)$ called the (smooth) *Whitehead space* of M, with $\pi_0 \mathbf{Wh}(M) = Wh(M)$.
- To every fiber bundle $B \to X$ as above, one can associate a map $\tau(B) : X \to \mathbf{Wh}(M)$ which is well-defined up to homotopy. When X is a point, the homotopy class of this map determines an element of $\pi_0 \mathbf{Wh}(M) \simeq Wh(M)$ which agrees with the classical Whitehead torsion of the bordism B.
- The map $\tau(B)$ is nullhomotopic whenever B is equivalent to a product $M \times [0,1] \times X$. One of our main objectives in this course will be to prove the following converse: if $\tau(B)$ is nullhomotopic, then the fiber bundles

$$B \times [0,1]^k \to X \qquad M \times [0,1]^{k+1} \times X$$

are equivalent for $k \gg 0$.

Meeting Time and Place MWF at 12, Science Center 310.

Office Hours Wednesday 2-3, or by appointment.

Texts Course notes will be provided on the course webpage. Some portions of this course will follow the book "Spaces of PL Manifolds and Categories of Simple Maps" by Waldhausen, Jahren, and Rognes. Pointers to relevant literature will be provided as the course proceeds.

Course Website http://www.math.harvard.edu/∼lurie/281.html

Prerequisites Familiarity with the machinery of modern algebraic topology (simplicial sets, spectra, ...). Several other topics (such as piecewise linear topology, microbundles, immersion theory, the language of quasi-categories) will receive a cursory review as we need them. A high level of mathematical sophistication will be assumed.

Possible Topics

- Wall's Finiteness Obstruction
- Simple homotopy equivalences and Whitehead torsion.
- Polyhedra and simple maps; "higher" simple homotopy theory
- Waldhausen's algebraic K-theory of spaces
- Assembly; the "parametrized index theorem" of Dwyer-Weiss-Williams.
- Regular neighborhood theory.
- Piecewise linear manifolds, microbundles, and immersion theory
- The parametrized (stable) s-cobordism theorem
- Concordance spaces and the Hatcher spectral sequence
- Hilbert cube manifolds and infinite-dimensional topology

Grading Undergraduates or graduate students wishing to take this course for a grade should speak with the instructor.