Fibrations of Polyhedra (Lecture 8)

September 20, 2014

In the previous lecture, we introduced a simplicial set $\mathcal{M}$ which parametrizes fibrations in the piecewise-linear category. To analyze $\mathcal{M}$, we consider the following:

Question 1. Let $q : E \to B$ be a piecewise-linear map of finite polyhedra. When is q a fibration?

To address this question, let us choose compatible triangulations τ_E and τ_B of E and B respectively, with vertex sets V_E and V_B. Then q maps each simplex of τ_E linearly onto a simplex of τ_B.

For each vertex $b \in V_B$, we let $E_b = q^{-1}\{b\}$ denote the fiber over b, so that τ_E induces a triangulation of E_b. More generally, suppose that σ is an n-simplex of τ_B with vertices $\{b_0, \ldots, b_n\}$. For every simplex $\overline{\sigma} \in \tau_E$ with $q(\overline{\sigma}) = \sigma$, let

$$\overline{\sigma}_i = \overline{\sigma} \cap E_{b_i}.$$

Let $E_\sigma \subseteq E_{b_0} \times \cdots \times E_{b_n}$ denote the union

$$\bigcup \overline{\sigma}_0 \times \cdots \times \overline{\sigma}_n$$

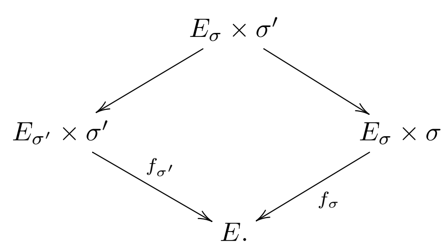
where the union is taken over all simplices $\overline{\sigma}$ of τ_E with $q(\overline{\sigma}) = \sigma$. Note that for every such $\overline{\sigma}$, there is a canonical map

$$f_\sigma : \overline{\sigma}_0 \times \cdots \times \overline{\sigma}_n \times \Delta^n \to \overline{\sigma} \subseteq E$$

given by

$$(x_0, \ldots, x_n, t_0, \ldots, t_n) \mapsto \sum t_i x_i.$$

Note that if $\sigma' \subseteq \sigma$ in τ_B, then there is a canonical projection map $E_\sigma \to E_{\sigma'}$ which fits into a commutative diagram



Exercise 2. The preceding maps can be assembled to a homeomorphism of topological spaces

$$\varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \simeq E.$$

In other words, the polyhedron E can be realized as the *coend* of the contravariant functor $\sigma \mapsto E_\sigma$ (from the partially ordered set τ_B to topological spaces) against the covariant functor $\sigma \mapsto \sigma$ (from τ_B to topological spaces).

Alternatively, this result can be interpreted as saying that the polyhedron E is the homotopy colimit of the functor $\sigma \mapsto E_\sigma$.

Warning 3. The homeomorphism

$$\varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \simeq E.$$

is generally not piecewise-linear (in fact, the colimit on the right hand side generally does not exist in the category of polyhedra). This is visible in the definition of the maps f_σ: the construction $(x_0, \ldots, x_n, t_0, \ldots, t_n) \mapsto \sum t_i x_i$ is quadratic, not linear.

Remark 4. It follows from Exercise 2 that for every point $b \in B$, the fiber $E_b = q^{-1}\{b\}$ is homeomorphic to E_σ where σ is the unique simplex of τ_B whose interior contains b.

It follows from Exercise 2 that E can be recovered (as a topological space) from the triangulation τ_B and the contravariant functor $\sigma \mapsto E_\sigma$. We can therefore address Question 1 as follows:

Theorem 5. *Let $q : E \to B$ be as above. Then q is a fibration if and only if, for every inclusion $\sigma' \subseteq \sigma$ in τ_B, the induced map $E_\sigma \to E_{\sigma'}$ is a cell-like map.*

The "only if" direction we have already proven: for each of the maps $\rho : E_\sigma \to E_{\sigma'}$, the mapping cylinder $M(\rho)$ can be realized as the fiber product $[0,1] \times_B E$ (where the path $[0,1] \to B$ is any straight line joining a point in the interior of σ to a point in the interior of σ'), so that if q is a fibration then $M(\rho) \to [0,1]$ is also a fibration and therefore ρ is cell-like. Our goal in this lecture is to prove the converse, following the argument given in [1].

Remark 6. In the previous lecture, we asserted without proof that a map of finite polyhedra $q : E \to B$ is a fibration if and only if it is a fibration over each simplex of a triangulation τ_B of B. This follows immediately from Theorem 5 (since we can always pass to a refinement of τ_B for which there is a compatible triangulation of E).

The other direction Theorem 5 is an immediate consequence of the following slightly more general statement:

Theorem 7. *Let B be a finite polyhedron with a triangulation τ_B. Suppose we are given a contravariant functor $\sigma \mapsto E_\sigma$ from τ_B to topological spaces. Assume that each E_σ is a compact ANR (for example, a finite polyhedron) and that each inclusion $\sigma' \subseteq \sigma$ induces a cell-like map $E_\sigma \to E_{\sigma'}$. Then the canonical map*

$$\mathrm{hocolim}_{\sigma \in \tau_B} E_\sigma = \varinjlim_{\sigma' \subseteq \sigma} E_\sigma \times \sigma' \to B$$

is a fibration.

Our proof will proceed by induction on the dimension of the polyhedron B. Recall that the condition that q be a fibration can be tested locally on B. Consequently, it will suffice to show that every point $b \in B$ has an open neighborhood U for which the induced map $E \times_B U \to U$ is a fibration. Passing to a subdivision of τ_B, we can arrange that b is a vertex of B.

Recall that the *closed star* C of b is the union of those simplices of τ_B which contain the vertex b, and the *link* L of b is the union of those simplices of τ_B which are contained in C but do not contain b. Then C can be identified with the cone $(L \times [0,1]) \amalg_{L \times \{1\}} *$, and the open star $C - L \simeq (L \times (0,1]) \amalg_{L \times \{1\}} *$ is an open neighborhood of b.

For each simplex σ of L, let $\sigma^+ \subseteq C$ denote the simplex spanned by σ and b, and set $E' = \mathrm{hocolim}_{\sigma \subseteq L} E_{\sigma^+}$. Since the link L has dimension smaller than the dimension of B, it follows from the inductive hypothesis that the canonical map $E' \to L$ is a fibration. Moreover, the maps $E_{\sigma^+} \to E_b$ assemble to give a cell-like map $E' \to \mathrm{hocolim}_{\sigma \subseteq L} E_b = L \times E_b$ of spaces fibered over L. Unwinding the definitions, we see that the fiber product $(C - L) \times_B E$ is homeomorphic to

$$(E' \times (0,1]) \amalg_{E' \times \{1\}} E_b.$$

It will therefore suffice to prove the following:

Proposition 8. *Let L be a finite polyhedron. Suppose we are given compact ANRs X and Y and a cell-like map $X \to Y \times L$ which induces a fibration $X \to L$. Then the induced map*

$$(X \times [0,1]) \amalg_{X \times \{1\}} Y \to (L \times [0,1]) \amalg_{L \times \{1\}} \{*\} = C(L)$$

is also a fibration.

The main ingredient we will need is the following lemma, which we will prove at the end of this lecture:

Lemma 9. *In the situation of Proposition 8, let $F : (X \times [0,1]) \amalg_{X \times \{1\}} Y \to Y \times C(L)$ be the canonical map. Then there exists a map $G : Y \times C(L) \to (X \times [0,1]) \amalg_{X \times \{1\}} Y$ and a homotopy H from the identity to $G \circ F$ which preserves fibers over $C(L)$ and is the identity on Y.*

Set $Z = (X \times [0,1]) \amalg_{X \times \{1\}} Y$. Proposition 8 asserts that every lifting problem of the form

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\overline{p}_0} & Z \\ \downarrow & \overset{\overline{p}}{\nearrow} & \downarrow \phi \\ A \times [0,1] & \xrightarrow{p} & C(L) \end{array}$$

has a solution. In this case, $\overline{p}_0$ determines a point $x \in Z$. There are two cases to consider.

Case (1) We have $p(0) = *$, so that x belongs to the fiber $\phi^{-1}\{*\} = Y$. In this case, we can define $\overline{p}$ by the formula $\overline{p}(t) = G(x, p(t))$.

Case (2) We have $p(0) \neq *$. In this case, there is a real number $0 < \theta \leq 1$ such that p carries the interval $[0, \theta]$ into the open set $C(L) - \{*\}$. We know ϕ restricts to a fibration $X \times [0,1) \to C(L) - \{*\}$, so that $p|_{[0,\theta]}$ can be lifted to a path $\overline{p}' : [0,\theta] \to X \times [0,1) \subseteq Z$. We can then define $\overline{p}$ by the formula

$$\overline{p}(t) = \begin{cases} H(\overline{p}'(t), \frac{t}{\theta}) & \text{if } t \leq \theta \\ G(\overline{p}(\theta), p(t)) & \text{if } t \geq \theta. \end{cases}$$

Let us now consider the case of a general parameter space A. We may assume without loss of generality that A is a metric space (in fact, it suffices to treat the universal case where $A = Z \times_{C(L)} C(L)^{[0,1]}$, which is metrizable). Let $A_1 = \{a \in A : p(a, 0) = *\}$ and let $A_2 = A - A_1$. We would like to construct the map $\overline{p}$ by applying the preceding recipes separately to A_1 and A_2. The main difference is that we will not regard θ as a constant, but instead as a continuous function $\theta : A_2 \to (0, 1]$. We will arrange that θ has the following property:

(a) We have $p(a,t) \neq *$ for $a \in A_2$ and $t \leq \theta(a)$.

Assume that θ satisfies (a) and define $B_\theta = \{(a,t) \in A_2 \times [0,1] : t \leq \theta(a)\}$. Using the fact that the map $X \times [0,1) \to C(L) - \{*\}$ is a fibration), we can extend $\overline{p}_0$ to a partially defined homotopy $\overline{p}'_\theta : B_\theta \subseteq Z$ extending $\overline{p}_0|_{A_2}$ and lying over $p|_{B_\theta}$. We can then define maps

$$\phi_1 : A_1 \times [0,1] \to Z \qquad \phi_2 : A_2 \times [0,1] \to Z$$

by the formulae

$$\phi_1(a,t) = G(\overline{p}_0(a), p(t))$$

$$\phi_2(a,t) = \begin{cases} H(\overline{p}'_\theta(a,t), \frac{t}{\theta(a)}) & \text{if } t \leq \theta(a) \\ G(\overline{p}'_\theta(a,t), p(a,t)) & \text{if } t \geq \theta(a). \end{cases}$$

Let $\overline{p} : A \times [0,1] \to Z$ be the map given on $A_i \times [0,1]$ by ϕ_i. To complete the proof, it will suffice to show that θ and $\overline{p}'_\theta$ can be chosen so that $\overline{p}$ is continuous.

Let us now assume:

(*b*) The map θ extends to a continuous map $\overline{\theta} : A \to [0,1]$ with $\overline{\theta}|_{A_1} = 0$.

Remark 10. To choose the map $\overline{\theta}$, let us equip the product $A \times [0,1]$ with the taxi-cab metric, and set $K = p^{-1}\{*\} \subseteq A \times [0,1]$. We can then define $\overline{\theta}$ by the formula

$$\overline{\theta}(a) = \min\{1, \frac{1}{2}d((a,0), K)\}.$$

Let $\overline{B}_\theta^{\leq} = \{(a,t) \in A \times [0,1] : t \leq \overline{\theta}(a)\}$, let $\overline{B}_\theta^{\geq} = \{(a,t) \in A \times [0,1] : t \geq \overline{\theta}(a)\}$. Assume that $\overline{p}|\overline{B}_\theta^{\leq}$ is continuous. Then the map $h : A \to Z$ given by $h(a) = \overline{p}(a, \overline{\theta}(a))$ is continuous. The restriction of $\overline{p}$ to $\overline{B}_\theta^{\geq}$ is given by

$$\overline{p}(a,t) = G(h(a), p(a,t)),$$

and is therefore continuous. It follows that $\overline{p}$ is continuous, as desired. We are therefore reduced to proving that we can choose $\overline{\theta}$ so that $\overline{p}|\overline{B}_\theta^{\leq}$ is continuous.

Note the map $(a,t) \mapsto (a, \overline{\theta}(a)t)$ determines a proper surjection $\pi : A \times [0,1] \to \overline{B}_\theta^{\leq}$. Consequently, it will suffice to show that $\overline{p} \circ \pi : A \times [0,1] \to Z$ is continuous. Let $r : A \times [0,1] \to Z$ be the map given by

$$r(a,t) = \begin{cases} \overline{p}_0(a) & \text{if } a \in A_1 \\ \overline{p}'(a, t\theta(a)) & \text{if } a \in A_2. \end{cases}$$

Then $(\overline{p} \circ \pi)(a,t) = H(r_\theta(a,t), t)$. It will therefore suffice to show that we can arrange that r is continuous. To prove this, let us choose a metric d_Z on the space Z and define $K' = \{(a,t) \in B_\theta : d_Z(\overline{p}'_\theta(a,t), \overline{p}_0(a)) \geq \theta(a)\}$. Let $\theta' : A_2 \to [0,1]$ be defined by the formula

$$\theta'(a) = \min\{\theta(t), d((a,0), K')\}.$$

Then $\theta' \leq \theta$, so θ' also extends to a continuous map $\overline{\theta}' : A \to [0,1]$ satisfying $\overline{\theta}'|_{A_1} = 0$. Replacing θ by θ' and $\overline{p}'_\theta$ with $\overline{p}'_\theta|_{B_{\theta'}}$, we can assume that the function r satisfies $d_Z(r(a,t), r(a,0)) \leq \theta(a)$ for $a \in A_2$. It follows that if we are given a sequence of points (a_i, t_i) in $A_2 \times [0,1]$ which approach a limit (a,t) in $A_1 \times [0,1]$, then we have

$$\lim r(a_i, t_i) = \lim r(a_i, 0) = r(a, 0) = r(a,t),$$

so that r is continuous as desired.

It remains to prove Lemma 9. The proof will require some careful estimates. From this point forward, we will fix a metric d on Y. We will employ the following abuse of notation: given any space Y' with a map $\pi : Y' \to Y$ and any pair of points $a, b \in Y'$, we set $d(a,b) = d(\pi(a), \pi(b))$ (note that this is generally *not* a metric on Y'; for example, points belonging to the same fiber of π have distance zero from one another). The cases of interest to are $Y' = Y \times L$ and $Y' = X$.

Given a map $Y' \to Y$ is as above, we will say that a path $p : [0,1] \to Y'$ is *ϵ-small* if, for every pair $s, t \in [0,1]$, the distance $d(p(s), p(t))$ is less than ϵ. More generally, given a topological space S and a homotopy $h : S \times [0,1] \to Y'$, we will say that h is *ϵ-small* if the paths $h|_{\{s\} \times [0,1]}$ are ϵ-small for each $s \in S$.

Let $f : X \to Y \times L$ denote the projection map. The main technical ingredient we will need is the following:

Proposition 11. *For each $\epsilon > 0$, there exists a map $g_\epsilon : Y \times L \to X$ and an ϵ-small homotopy h_ϵ : $X \times [0,1] \to X$ from id_X to $g_\epsilon \circ f$, where g_ϵ and h_ϵ are compatible with projection to L.*

Let us assume Proposition 11 for the moment and show that it leads to a proof of Lemma 9.

Remark 12. In the situation of Proposition 11, it follows from the existence of the homotopy h_ϵ that we have $d(y, (f \circ g_\epsilon)(y)) < \epsilon$ for each $y \in Y \times L$. In other words, the maps g_ϵ are *approximately* sections to f.

Remark 13. We have assumed that Y is a compact ANR, so there exists an embedding of Y into a Banach space B and a retraction $r : U \to Y$, where U is an open neighborhood of Y in B. Fix a real number $\epsilon > 0$. For sufficiently small δ, any pair of points $y, y' \in Y$ with $d(y, y') < \delta$ have the property that the interval joining y to y' in B belongs entirely to U, so that the construction

$$p_{y,y'} : [0,1] \to Y$$

$$p_{y,y'}(t) = r((1-t)y + ty')$$

determines a continuous path from y to y' in Y. The path $p_{y,y'}$ depends continuously on y and y'. It follows that the function $(y, y') \mapsto \sup\{d(p_{y,y'}(s), p_{y,y'}(t))\}$ is also a continuous function, which vanishes when $y = y'$. Shrinking δ if necessary, we may assume that $\delta < \epsilon$ and that if $d(y, y') < \delta$ then the path $p_{y,y'}$ is ϵ-small. In this case, we will say that δ *is small compared to* ϵ and write $\delta \ll \epsilon$.

Remark 14. Suppose that $\delta \ll \epsilon$. It follows from Remark 12 that for each $y \in Y \times L$, there is an ϵ-small path joining y to $(f \circ g_\delta)(y)$, which depends continuously on y. These paths can be assembled to a $k_{\delta,\epsilon} : Y \times L \times [0,1] \to Y \times L$. which is compatible with the projection to L. In particular, we see that g_δ is a right homotopy inverse to f (it is also a left homotopy inverse, by virtue of the existence of the homotopy h_ϵ.

Remark 15. Suppose that $2\delta \ll \epsilon$. For every point $x \in X$, the constructions

$$t \mapsto F(h_\delta(x,t)) \qquad t \mapsto k_{\delta,\epsilon}(f(x), t)$$

determine δ-small paths from $f(x)$ to $(f \circ g_\delta \circ f)(x)$. Using the triangle inequality, we see that the distance between these paths is at most 2δ. It follows that there is an ϵ-small homotopy

$$v_{\delta,\epsilon} : X \times [0,1] \times [0,1] \to Y \times L$$

from $F \circ h_\delta$ to $k_{\delta,\epsilon} \circ (F \times \mathrm{id}_{[0,1]})$.

We are now ready to construct the map G appearing in the statement of Lemma 9. Choose a sequence of positive real numbers $\epsilon_0, \epsilon_1, \ldots$ with $\epsilon_0 \leq 1$ and $2\epsilon_{n+1} \ll \epsilon_n$ (from which it follows that $\epsilon_n \leq \frac{1}{2^n}$ for all n). We define a continuous map $G^\circ : Y \times L \times \mathbb{R}_{\geq 0} \to X$ by the formula

$$G^\circ(y,t) = \begin{cases} g_{\epsilon_n}(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) & \text{if } n \leq t \leq n + \frac{1}{2} \\ h_{\epsilon_n}(g_{\epsilon_{n+1}}(y), 2(n+1-t)) & \text{if } n + \frac{1}{2} \leq t \leq n+1. \end{cases}.$$

Let us identify $\mathbb{R}_{\geq 0}$ with the half-open interval $[0,1)$, so that the construction

$$(y,t) \mapsto (G^\circ(y), t)$$

determines a continuous map $Y \times L \times [0,1) \to X \times [0,1)$. We claim that map admits a continuous extension $G : Y \times L \times [0,1] \to Z$ whose restriction to $Y \times L \times \{1\}$ is given by the projection onto the first factor. To prove this, it suffices to show that for every sequence of points (y_i, t_i) in $Y \times L \times [0, \infty)$ where the y_i converge to some point $y \in Y \times L$ and the t_i converge to ∞, the sequence of points $f(G^\circ(y_i, t_i))$ converge to y in $Y \times L$. This is clear: note that if $n \leq t \leq n + \frac{1}{2}$, then

$$\begin{aligned} d(y, f(G^\circ(y,t))) &\leq d(y, k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) + d(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))), (f \circ g_{\epsilon_n})(k_{\epsilon_{n+1},\epsilon_n}(y, 2(t-n))) \\ &\leq 2\epsilon_n, \end{aligned}$$

while for $n + \frac{1}{2} \leq t \leq n+1$ we have

$$\begin{aligned} d(y, f(G^\circ(y,t))) &\leq d(y, f(G^\circ(y, n+1))) + d(f(G^\circ(y, n+1)), f(G^\circ(y,t)) \\ &= d(y, fg_{\epsilon_{n+1}}(y)) + d(f(G^\circ(y, n+1)), f(G^\circ(y,t)) \\ &\leq \epsilon_{n+1} + \epsilon_n. \end{aligned}$$

To construct the homotopy H, we begin by considering a map $T : X \times [0, \infty) \to X$ given by the formula

$$T(x,t) = \begin{cases} g_{\epsilon_n}(f(h_{\epsilon_{n+1}}(x, 2(t-n)))) & \text{if } n \le t \le n + \frac{1}{2} \\ h_{\epsilon_n}(g_{\epsilon_{n+1}}(f(x)), 2(n+1-t)) & \text{if } n + \frac{1}{2} \le t \le n+1. \end{cases}$$

There is a canonical homotopy from the projection map $\pi : X \times [0,\infty) \to X$ to T, which carries a pair $(x,t) \in X \times [0,\infty)$ to the path

$$s \mapsto \begin{cases} h_{\epsilon_n}(h_{\epsilon_{n+1}}(x, 2(t-n)s), s) & \text{if } n \le t \le n + \frac{1}{2} \\ h_{\epsilon_n}(h_{\epsilon_{n+1}}(x, s), 2(n+1-t)s) & \text{if } n + \frac{1}{2} \le t \le n+1. \end{cases}$$

The maps $v_{\epsilon_{n+1},\epsilon_n}$ of Remark 15 can be assembled to a homotopy from T to the map $(x,t) \mapsto G(f(x),t)$. Concatenating these homotopies, we obtain a map

$$H^\circ : X \times [0,\infty) \times [0,1] \to X \times [0,\infty).$$

We claim that (after identifying $[0,\infty)$ with $[0,1)$) H° extends continuously to a homotopy $H : Z \times [0,1] \to Z$ from id_Z to G which is trivial on the closed subset $Y \subseteq Z$. To prove this, we must show that if $\{x_i\}$ is a sequence of points of X whose images in Y converge to a point y and $\{t_i\}$ is a sequence of positive real numbers which converges to ∞, then the paths $(\pi_Y \circ f \circ H)|_{\{(x_i,t_i)\} \times [0,1]}$ converge to the constant path based at the point y, which is a consequence of the following elementary lemma which we leave to the reader:

Lemma 16. *Let $\{p_i : [0,1] \to Y\}_{i \ge 0}$ be a sequence of continuous paths in Y. Assume that:*

- (a) *For each $\epsilon > 0$, the paths p_i are ϵ-small for almost all i.*
- (b) *The sequence of points $\{p_i(0)\}_{i \ge 0}$ converges to a point $y \in Y$.*

Then the paths p_i converge to the constant path $[0,1] \to \{y\} \hookrightarrow Y$.

We now turn to the proof of Proposition 11. In the case where L is a single point, we have the following:

Proposition 17. *Let $f : X \to Y$ be a surjective map of compact ANRs. The following conditions are equivalent:*

1. *The map f is cell-like.*
2. *For every $\epsilon > 0$, there exists a map $g : Y \to X$ and an ϵ-small homotopy $h : X \times [0,1] \to X$ from the identity map to $g \circ f$ (recall that all distances are measured with respect to some metric on Y).*

Let us assume Proposition 17 for a moment, and see how it leads to a proof of Proposition 11. Choose a metric on L. Given a cell-like map $f : X \to Y \times L$ and any $\epsilon > 0$, Proposition 17 guarantees the existence of a map $g' : Y \times L \to X$ and a homotopy $h' : X \times [0,1] \to X$ from id_X to $g' \circ f$ such that the homotopy $f \circ h'$ is ϵ-small both in Y *and* in L. We wish to show that we can arrange that g' and h' commute with the projection to L. We will deduce this from the following:

Lemma 18. *Fix $\delta > 0$. For each $\epsilon > 0$, let $U \subseteq X \times L$ be the open set consisting of those points (x,v) such that the distance from v to the image of x (measured with respect to the metric on L) is $< \epsilon$. For ϵ sufficiently small, there exists a map $r : U \to X$ satisfying the following conditions:*

1. *The map r commutes with the projection to L.*
2. *If $x \in X$ and v is its image in L, then $r(x,v) = x$.*
3. *For all $(x,v) \in X$, there is an δ-small path from $r(x,v)$ to x.*

If ϵ is chosen small enough to satisfy the requirements of Lemma 18, then we can set

$$g(y) = r(g'(y), \pi_L g'(y)) \qquad h(x,t) = r(h'(x,t), \pi_L h'(x,t))$$

where $\pi_L : X \to L$ is the projection map. It then follows from the triangle inequality that the homotopy h is $(\epsilon + 2\delta)$-small; Proposition 11 then follows choosing δ and ϵ sufficiently small.

Proof of Lemma 18. Since the map $\pi_L : X \to L$ is a fibration, we can choose a path lifting function $u : X \times_L L^{[0,1]} \to X^{[0,1]}$. Let us identify X with its image in $X \times_L L^{[0,1]}$ (that is, the set of pairs (x, c) where $c : [0,1] \to L$ is the constant path based at $\pi_L(x)$). Without loss of generality, we can assume that the restriction $u|_X$ is the diagonal embedding $X \hookrightarrow X^{[0,1]}$.

Applying the discussion of Remark 13 to the space L, we see that if ϵ is sufficiently small, then any two points $v, v' \in L$ at distance $< \epsilon$ can be joined by a path $p_{v,v'}$ which depends continuously on v and v'. We can then define $r : U \to X$ by the formula

$$r(x, v) = u(x, p_{\pi_L(x),v})(1).$$

It is easy to see that r satisfies conditions (1) and (2), and condition (3) can be ensured by shrinking ϵ if necessary. □

Proof of Proposition 17. We first show that (2) ⇒ (1) (we don't actually need this implication, but it is a pleasant characterization of the class of cell-like maps). We will show that each fiber X_y of f has trivial shape. Since X_y is nonempty, it suffices to show that for any CW complex S, any map $f_0 : X_y \to S$ is nullhomotopic. The map f_0 extends continuously to a map $f : V \to S$, where V is some open neighborhood of X_y in X. It will therefore suffice to show that the inclusion $X_y \hookrightarrow V$ is nullhomotopic. Choose ϵ small enough that $V \subseteq f^{-1} B_\epsilon(y)$, where $B_\epsilon(y)$ denotes a ball of radius ϵ about Y. Assumption (2) implies that there exists a map $g : Y \to X$ and an ϵ-small homotopy from id_X to $g \circ f$. This restricts to a homotopy from the inclusion map $X_y \hookrightarrow V$ to a constant map.

We now consider the interesting direction: the implication (1) ⇒ (2). Since Y is compact, we can cover Y by finitely many balls of radius ϵ; let us denote those balls by $\{U_i\}_{i \in I}$. For every nonempty subset $J \subseteq I$, set $U_J = \bigcap_{i \in J} U_i$. For every chain $J_0 \subseteq \cdots \subseteq J_m$ of nonempty subsets of I, we will construct a map

$$G_{\vec{J}} : \Delta^m \times U_{J_m} \to f^{-1} U_{J_0}$$

and a homotopy

$$H_{\vec{J}} : \Delta^m \times f^{-1} U_{J_m} \to f^{-1} U_{J_0}$$

from the identity to $G_{\vec{J}} \circ f$. Moreover, we will choose these maps to be compatible with one another in the sense that if $\vec{J}' = (J'_0 \subseteq \cdots \subseteq J'_{m'})$ is another chain of nonempty subsets of I which is contained in $\vec{J}$, then $H_{\vec{J}}$ and $H_{\vec{J}'}$ agree on $\Delta^{m'} \times f^{-1} U_{J_m}$ (which implies that $G_{\vec{J}}$ and $G_{\vec{J}'}$ agree on $\Delta^{m'} \times U_{J_m}$). The construction proceeds by induction on the size of $\vec{J}$; at each stage, we are forced to extend a map over the inclusion

$$i : (\partial \Delta^m \times M) \amalg_{(\partial \Delta^m \times f^{-1} U_{J_m})} (\Delta^m \times f^{-1} U_{J_m}) \hookrightarrow \Delta^m \times M$$

where M denotes the mapping cylinder of the projection $f^{-1} U_{J_m} \to U_{J_m}$. To show that this extension is possible, it suffices to show that i admits a left inverse. This follows from the fact $f^{-1} U_{J_m}$ is a deformation retract of M (since the projection $f^{1} U_{J_m} \to U_{J_m}$ is a homotopy equivalence by virtue of our assumption that f is cell-like).

Let P denote the partially ordered set of nonempty subsets of I and let Δ denote the nerve of P. Then Δ is a topological simplex with vertices corresponding to the elements of I, and its presentation as the nerve of P gives a triangulation of Δ (given by barycentric subdivision) with one m-simplex $\sigma_{\vec{J}}$ for every chain $\vec{J} = (J_0 \subseteq \cdots \subseteq J_m)$ as above.

Choose a partition of unity $\{\lambda_i\}_{i \in I}$ on Y having the property that for each index i, the *closure* of the support of λ_i is contained in the open set U_i. We can regard the λ_i as defining a continuous map $\lambda : Y \to \Delta$.

Moreover, for each $y \in Y$ there exists a chain $\vec{J} = (J_0 \subseteq \cdots \subseteq J_m)$ such that $\lambda(y) \in \sigma_{\vec{J}} \simeq \Delta^m$ and $y \in U_{J_m}$. We define $G : Y \to X$ by the formula $G(y) = G_{\vec{J}}(y, \lambda(y))$. Similarly, for $x \in X$ with $f(x) = y$, we set $H(x, t) = H_{\vec{J}}(x, \lambda(y), t)$. It is not difficult to see that G and H are well-defined and have the desired properties. □

References

[1] Hatcher, A. *Higher Simple Homotopy Theory.*