Concordance of Polyhedra (Lecture 6)

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We begin by tying up a few loose ends from the previous lecture. We begin with a few remarks about cell-like maps:

Example 1. Let $f : X \to Y$ be a proper map of Hausdorff spaces, and let $M(f) = (X \times [0,1]) \amalg_{X \times \{1\}} Y$ be its mapping cylinder. Then the canonical retraction $r : M(f) \to Y$ (given on $X \times [0,1]$ by $(x,t) \mapsto f(x)$) is a cell-like map: the inverse image of each point $y \in Y$ homeomorphic to the cone on $f^{-1}\{y\}$, and therefore contractible.

Example 2. Let Y be a CW complex, and suppose we are given a cellular map $f : D^n \to Y$. Then the mapping cylinder $M(f) = (D^n \times [0,1]) \amalg_{D^n \times \{1\}} Y$ is an elementary expansion of Y, and the map $M(f) \to Y$ of Example 1 is the associated elementary collapse. It follows that any elementary collapse is a cell-like map. Consequently, any simple homotopy equivalence of CW complexes can be obtained by composing cell-like maps and their homotopy inverses.

Warning 3. In the previous lecture, we showed that a proper map $f : X \to Y$ of CW complexes is cell-like if and only if, for each $U \subseteq Y$, the induced map $f^{-1}U \to U$ is a homotopy equivalence. It follows that the collection of cell-like maps between CW complexes is closed under composition. Beware that the collection of cell-like maps is not closed under composition in general (one can avoid this problem by slightly modifying the definition of cell-like map, but this will not be important for us in what follows).

We now recall the result promised in the previous lecture:

Proposition 4. *Let $f : X \to Y$ be a map of finite CW complexes. Assume that f is cell-like, cellular, and that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

Example 5. Let $f : X \to Y$ be a piecewise linear map of polyhedra. Then one can always choose triangulations τ_X and τ_Y of X and Y respectively that are *compatible with* f in the following sense:

(1) The map f carries each vertex of the triangulation τ_X to a vertex of the triangulation τ_Y.

(2) The map f is linear on each simplex of the triangulation τ_X (and therefore maps each simplex of τ_X linearly onto a simplex of τ_Y).

The triangulations τ_X and τ_Y determine CW structures on X and Y for which the induced map $f : X \to Y$ is cellular, and for which the inverse image $f^{-1}e$ is a union of cells for each cell $e \subseteq Y$. Consequently, any cell-like piecewise-linear map of polyhedra $f : X \to Y$ can be considered to satisfy the requirements of Proposition 4.

Proof of Proposition 4. Without loss of generality we may assume that Y is connected. The map f is a homotopy equivalence by Proposition **??**. To show that it is a simple homotopy equivalence, it will suffice to show that the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1 Y)$ vanishes.

Let $\widetilde{Y}$ be a universal cover of Y, and for every space Z with a map $Z \to Y$ we let $\widetilde{Z}$ denote the fiber product $Z \times_Y \widetilde{Y}$ (this is a covering space of Z which may or may not be connected). Set $G = \pi_1 Y$ so that

G acts on $\widetilde{Y}$ by deck transformations. We wish to show that the torsion of the quasi-isomorphism of cellular chain complexes $C_*(\widetilde{X};\mathbf{Z}) \to C_*(\widetilde{Y};\mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$. We will deduce this from the following more general assertion:

($*$) For every subcomplex $Y_0 \subseteq Y$ with inverse image $X_0 \subseteq X$, the torsion of the induced map of cellular chain complexes $C_*(\widetilde{X}_0;\mathbf{Z}) \to C_*(\widetilde{Y}_0;\mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$.

We proceed by induction on the number of cells in Y_0. Let us therefore assume that ($*$) is known for a subcomplex $Y_0 \subseteq Y$, and see what happens when we add one more cell to obtain another subcomplex $Y_1 \subseteq Y$. As we saw in the previous lecture, Whitehead torsion is multiplicative in short exact sequences. To carry out the inductive step, it will suffice to show that torsion of the map of relative cellular cochain complexes

$$\theta : C_*(\widetilde{X}_1, \widetilde{X}_0;\mathbf{Z}) \to C_*(\widetilde{Y}_1, \widetilde{Y}_0;\mathbf{Z})$$

vanishes in $\mathrm{Wh}(G)$. Let $e \subseteq Y$ be the open cell of Y_1 which does not belong to Y_0. Then e is simply connected, so the map $\widetilde{e} \to e$ admits a section. A choice of section (and orientation of e) determines a one-element basis for $C_*(\widetilde{Y}_1, \widetilde{Y}_0;\mathbf{Z})$ as a $\mathbf{Z}[G]$-module, and by choosing cells of $\widetilde{X}_1 - \widetilde{X}_0$ which belong to the inverse image of the section we obtain a basis for $C_*(\widetilde{X}_1, \widetilde{X}_0;\mathbf{Z})$ as a $\mathbf{Z}[G]$-module (ambiguous up to signs) with respect to which the differential on $C_*(\widetilde{X}_1, \widetilde{X}_0;\mathbf{Z})$ and the map θ can be represented by matrices with integral coefficients (rather than coefficients in $\mathbf{Z}[G]$). It follows that the image of $\tau(\theta)$ in $\mathrm{Wh}(G)$ factors through the map $\widetilde{K}_1(\mathbf{Z}) = \mathrm{Wh}(*) \to \mathrm{Wh}(G)$, and therefore vanishes because the group $\widetilde{K}_1(\mathbf{Z}) = K_1(\mathbf{Z})/\{[\pm 1]\}$ is trivial. □

Recall that a map of topological spaces $q : E \to B$ is said to be a *fibration* if it has the homotopy lifting property: that is, for any map $f : X \to E$ and any homotopy h from $(q \circ f)$ to another map $g : X \to B$, there exists a homotopy $\overline{h} : X \times [0,1] \to E$ with $\overline{h}|_{X \times \{0\}} = f$ and $q \circ \overline{h} = h$.

Let $q : E \to B$ be a fibration, and let $p : [0,1] \to B$ be a continuous path which begins at a point $b = p(0)$ and ends at a point $b' = p(1)$. The construction

$$(e,t) \mapsto p(t)$$

determines a homotopy from the constant map $E_b \to \{b\} \subseteq B$ to the constant map $E_b \to \{b'\} \subseteq B$. Applying the homotopy lifting property, we deduce that there is a continuous map $\overline{h} : E_b \times [0,1] \to E$ such that $q(\overline{h}(e,t)) = p(t)$. The restriction $\overline{h}|_{E_b \times \{1\}}$ is a continuous map from E_b to $E_{b'}$. We will denote this map by $p_! : E_b \to E_{b'}$; one can show that up to homotopy, it depends only on the homotopy class of the path p (and not on the choice of map $\overline{h}$). The map $p_!$ is always a homotopy equivalence: it has a homotopy inverse given by $p'_!$, where $p' : [0,1] \to B$ is given by $p'(t) = p(1-t)$.

In the special case where $B = [0,1]$ and we take p to be the identity map, we see that any fibration $q : E \to [0,1]$ determines a homotopy equivalence $p_! : E_0 \to E_1$. Our objective in this course is to study what further constraints on $p_!$ are imposed by the condition that the total space E be "finite" in some sense.

Definition 6. Let X and Y be finite polyhedra. A *concordance* from X to Y is a (piecewise linear) fibration of finite polyhedra $q : E \to [0,1]$ together with PL homeomorphisms $X \simeq q^{-1}\{0\}$ and $Y \simeq q^{-1}\{1\}$. We will say that X and Y are *concordant* if there exists a concordance from X to Y.

Note that every concordance from X to Y determines a homotopy equivalence $p_! : X \to Y$, which is well-defined up to homotopy. We can now state our next main result:

Theorem 7. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a homotopy equivalence. The following conditions are equivalent:*

(1) *The map f is a simple homotopy equivalence.*

(2) *There exists a concordance $q : E \to [0,1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

Corollary 8. *Let X and Y be finite polyhedra. Then X and Y are simple homotopy equivalent if and only if they are concordant.*

We will prove the implication (2) $\Rightarrow$ (1) in this lecture, saving the reverse implication for later. Let us suppose that we have chosen triangulations τ_E and τ_B of E and B which are compatible in the sense of Example **??**. The triangulation τ_B is just a partition of B into subintervals. We may therefore reduce to the case where the triangulation τ_B consists of only a single interval (and its endpoints 0 and 1).

For every simplex σ of τ_E which is not contained in either E_0 or E_1, we let $\sigma_0 = \sigma \cap E_0$ and $\sigma_1 = \sigma \cap E_1$. Let $E_{01} \subseteq E_0 \times E_1$ denote the subset given by those products $\sigma_0 \times \sigma_1$, where σ ranges over those simplices of τ_E which are contained in neither E_0 nor E_1. Then E_{01} is equipped with projection maps $h_0 : E_{01} \to E_0$ and $h_1 : E_{01} \to E_1$. By construction, for every point $x \in E_{01}$, the line segment joining $h_0(x)$ to $h_1(x)$ is contained in E. We can therefore define a "straight-line" homotopy $h : E_{01} \times [0,1] \to E$ by the formula

$$h(x,t) = (1-t)h_0(x) + th_1(x).$$

Exercise 9. Show that the homotopy h above induces a homeomorphism of topological spaces

$$E_0 \amalg_{E_{01}\times\{0\}} (E_{01} \times [0,1]) \amalg_{E_{01}\times\{1\}} E_1 \to E.$$

In other words, if $M(h_0)$ and $M(h_1)$ denote the mapping cylinders of h_0 and h_1 respectively, then E is homeomorphic to the pushout $M(h_0) \amalg_{E_{01}} M(h_1)$.

Warning 10. The homotopy h is usually *not* piecewise-linear. The description of E as a pushout $M(h_0) \amalg_{E_{01}} M(h_1)$ is valid only in the category of topological spaces, not in the category of polyhedra.

The above argument proves the following:

Proposition 11. *Let E be a finite polyhedron equipped with a piecewise linear map $E \to [0,1]$. Then it is possible to subdivide $[0,1]$ into finitely many subintervals such that the inverse image of each subinterval $[t,t']$ is homeomorphic either to the mapping cylinder of a PL map $E_t \to E_{t'}$ or to the (reversed) mapping cylinder of a PL map $E_{t'} \to E_t$.*

We will deduce the implication (2) $\Rightarrow$ (1) of Theorem 6 from Proposition 10 together with the following:

Proposition 12. *Let $f : X \to Y$ be a continuous map of compact Hausdorff spaces and let $M(f) = (X \times [0,1]) \amalg_{X\times\{1\}} Y$ denote the mapping cylinder of f. If the projection map $q : M(f) \to [0,1]$ is a fibration, then f is a cell-like map.*

We will later see that the converse of Proposition 11 is true as well, provided that the spaces X and Y are reasonably nice.

Proof of Proposition 11. Fix a point $y \in Y$. We wish to show that the fiber $X_y = f^{-1}\{y\}$ has trivial shape. In other words, we wish to show that every map from X_y to a CW complex Z is nullhomotopic. Any such map extends over a neighborhood $U \subseteq X$ containing X_y; it will therefore suffice to show that the inclusion $X_y \hookrightarrow U$ is nullhomotopic. Since f is a proper map, U contains an open set of the form $f^{-1}V$ where V is a neighborhood of y in Y. We may therefore assume without loss of generality that $U = f^{-1}V$.

Let $C \subseteq M(f)$ denote the mapping cylinder of the projection $X_y \to \{y\}$ and let $W \subseteq M(f)$ be the mapping cylinder of the projection $U \to V$. Since q is a fibration, we can choose a homotopy $h : C \times [0,1] \to M(f)$ such that $h(c,0) = c$ and $q(h(c,t)) = \max\{q(c), 1-t\}$. Since $h|_{C\times\{0\}}$ factors through W, it follows that there exists $\epsilon > 0$ such that $h|_{C\times[0,\epsilon]}$ factors through W. Let $g : X_y \to f^{-1}(V)$ be the map characterized by

$$h(x, 1-\epsilon, \epsilon) = (g(x), 1-\epsilon).$$

Restricting h to $(X_y \times \{1-\epsilon\}) \times [0,\epsilon]$, we see that g is homotopic to the inclusion $X_y \hookrightarrow f^{-1}(V)$. On the other hand, by restricting h to $((X_y \times [1-\epsilon,1]) \amalg_{X_y\times\{1\}} \{y\}) \times \{\epsilon\} \subseteq C \times [0,1]$, we see that g is nullhomotopic. It follows that the inclusion $X_y \hookrightarrow f^{-1}(V)$ is nullhomotopic, as desired. □