Cell-Like Maps (Lecture 5)

September 15, 2014

In the last two lectures, we discussed the notion of a *simple* homotopy equivalences between finite CW complexes. *A priori*, the question of whether or not a map $f : X \to Y$ is a simple homotopy equivalence depends on the specified cell decompositions of X and Y. This motivates the following:

Question 1. Let $f : X \to Y$ be a homeomorphism of finite CW complexes. Is f a simple homotopy equivalence?

Chapman gave an affirmative action to Question 1 using techniques of infinite-dimensional topology. The following special case is much easier:

Proposition 2. *Let $f : X \to Y$ be a homeomorphism of finite CW complexes with the property that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

We postpone the proof; we will discuss a stronger result at the end of this lecture. Let us instead discuss some applications of Proposition 2.

Definition 3. Let X be a topological space. A *simple homotopy structure* on X is an equivalence class of homotopy equivalences $f : X \to Y$ (where Y is a finite CW complex), where we regard $f : X \to Y$ and $f' : X \to Y'$ as equivalent if the composition $f' \circ g$ is a simple homotopy equivalence, where $g : Y \to X$ denotes a homotopy inverse to f.

If X and X' are topological spaces equipped with simple homotopy structures represented by homotopy equivalences $f : X \to Y$ and $f' : X' \to Y'$, then we will say that a map $h : X \to X'$ is a *simple homotopy equivalence* if the composition $f' \circ h \circ g : Y \to Y'$ is a simple homotopy equivalence, where g is a homotopy inverse to f.

Example 4. Let X be a topological space. Any cell decomposition which exhibits X as a finite CW complex determines a simple homotopy structure on X.

We now discuss some examples of topological spaces which can be endowed with simple homotopy structures in a natural way.

Definition 5. Let K be a subset of a Euclidean space $\mathbb{R}^n$. We will say that K is a *linear simplex* if it can be written as the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which are independent in the sense that if $\sum c_i x_i = 0 \in \mathbb{R}^n$ and $\sum c_i = 0 \in \mathbb{R}$, then each c_i vanishes (equivalently, if the convex hull of $\{x_1, \ldots, x_k\}$ has dimension exactly $(k-1)$).

We will say that K is a *polyhedron* if, for every point $x \in K$, there exists a finite number of linear simplices $\sigma_i \subseteq K$ such that the union $\bigcup_i \sigma_i$ contains a neighborhood of X.

Remark 6. Any open subset of a polyhedron in $\mathbb{R}^n$ is again a polyhedron.

Remark 7. Every polyhedron $K \subseteq \mathbb{R}^n$ admits a *PL triangulation*: that is, we can find a collection of linear simplices $S = \{\sigma_i \subseteq K\}$ with the following properties:

(1) Any face of a simplex belonging to S also belongs to S.

(2) Any nonempty intersection of any two simplices of S is a face of each.

(3) The union of the simplices σ_i is K.

Moreover, one can show that any two PL triangulations of K admit a common refinement.

Remark 8. Let X be a polyhedron. The following conditions are equivalent:

- As a topological space, X is compact.
- Every PL triangulation of X involves only finitely many simplices.
- There exists a PL triangulation of X involving only finitely many simplices.

If these conditions are satisfied, we will say that X is a *finite polyhedron*.

Example 9. Let X be a finite polyhedron. Then any PL triangulation of X determines a simple homotopy structure on X. By virtue of Remark 7 and Proposition 2, this simple homotopy structure is independent of the choice of PL triangulation.

Definition 10. Let $K \subseteq \mathbb{R}^n$ be a polyhedron. We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will say that f is *piecewise linear* (PL) if there exists a triangulation $\{\sigma_i \subseteq K\}$ such that each of the restrictions $f|_{\sigma_i}$ is linear.

If $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are polyhedra, we say that a map $f : K \to L$ is *piecewise linear* if the underlying map $f : K \to \mathbb{R}^m$ is piecewise linear.

Remark 11. Let $f : K \to L$ be a piecewise linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f, we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Remark 12. Any piecewise linear homeomorphism of polyhedra $f : X \to Y$ is a simple homotopy equivalence (with respect to the simple homotopy structures of Example 9).

Remark 13. The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ can be isomorphic even if $n \neq m$.

Example 14. Let M be a compact smooth manifold. Then M admits a *Whitehead triangulation*: that is, there is a finite simplicial complex K equipped with a homeomorphism $K \to M$ which is differentiable (with injective differential) on each simplex of K. This determines a simple homotopy structure on M which is *independent* of the choice of Whitehead triangulation (this follows from the fact that any pair of Whitehead triangulations $f : K \to M$ and $g : L \to M$, one can find arbitrarily close Whitehead triangulations $f' : K \to M$ and $g' : L \to M$ such that the homeomorphism ${g'}^{-1} \circ f' : K \to L$ is piecewise linear).

By virtue of Example 14, it makes sense to ask if a map between compact smooth manifolds is a simple homotopy equivalence (as in the statement of the s-cobordism theorem). Let us now turn to Proposition 2 itself.

Proof of Proposition 2. Without loss of generality we may assume that Y is connected. The assumption that $f^{-1}e$ is a union of cells for each open cell $e \subseteq Y$ implies that the inverse homeomorphism $f^{-1} : Y \to X$ is cellular. Let $\widetilde{Y}$ be a universal cover of Y, and for every space Z with a map $Z \to Y$ we let $\widetilde{Z}$ denote the fiber product $Z \times_Y \widetilde{Y}$ (this is a covering space of Z which may or may not be connected). Set $G = \pi_1 Y$ so that G acts on $\widetilde{Y}$ by deck transformations. We will prove that f is a simple homotopy equivalence by showing that f^{-1} induces a map of cellular chain complexes $C_*(\widetilde{Y}; \mathbf{Z}) \to C_*(\widetilde{X}; \mathbf{Z})$ with vanishing Whitehead torsion. We will deduce this from the following more general assertion:

(∗) For every subcomplex $Y_0 \subseteq Y$ with inverse image $X_0 \subseteq X$, the torsion of the induced map of cellular chain complexes $C_*(\widetilde{Y}_0; \mathbf{Z}) \to C_*(\widetilde{X}_0; \mathbf{Z})$ vanishes in $\mathrm{Wh}(G)$.

We proceed by induction on the number of cells in Y_0. Let us therefore assume that (∗) is known for a subcomplex $Y_0 \subseteq Y$, and see what happens when we add one more cell to obtain another subcomplex $Y_1 \subseteq Y$. As we saw in the previous lecture, Whitehead torsion is multiplicative in short exact sequences. To carry out the inductive step, it will suffice to show that torsion of the map of relative cellular cochain complexes

$$\theta : C_*(\widetilde{Y}_1, \widetilde{}_0; \mathbf{Z}) \to C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$$

vanishes in $\mathrm{Wh}(G)$. Let $e \subseteq Y$ be the open cell of Y_1 which does not belong to Y_0. Then e is simply connected, so the map $\widetilde{e} \to e$ admits a section. A choice of section (and orientation of e) determines a one-element basis for $C_*(\widetilde{Y}_1, \widetilde{Y}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module, and by choosing cells of $\widetilde{X}_1 - \widetilde{X}_0$ which belong to the inverse image of the section we obtain a basis for $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module (ambiguous up to signs) with respect to which the differential on $C_*(\widetilde{X}_1, \widetilde{X}_0; \mathbf{Z})$ and the map θ can be represented by matrices with integral coefficients (rather than coefficients in $\mathbf{Z}[G]$). It follows that the image of $\tau(\theta)$ in $\mathrm{Wh}(G)$ factors through the map $\widetilde{K}_1(\mathbf{Z}) = \mathrm{Wh}(*) \to \mathrm{Wh}(G)$, and therefore vanishes because the group $\widetilde{K}_1(\mathbf{Z}) = K_1(\mathbf{Z})/\{[\pm 1]\}$ is trivial. □

Our next goal is to prove an analogue of Proposition 2 where we weaken the hypothesis that f is a homeomorphism. First, we need to introduce some terminology.

Definition 15. Let X be a topological space. We will say that X has *trivial shape* if X is nonempty and, for every CW complex Y, every map $f : X \to Y$ is homotopic to a constant map.

Example 16. Any contractible topological space X has trivial shape. The converse holds if X has the homotopy type of a CW complex.

Let us collect up some consequences of Definition 15:

Proposition 17. *Let X be a paracompact topological space with trivial shape. Then:*

(1) *The space X is connected.*

(2) *Every locally constant sheaf (of sets) on X is constant.*

(3) *For every abelian group A, we have*

$$\mathrm{H}^*(X; \underline{A}) \simeq \begin{cases} A & if * = 0 \\ 0 & otherwise. \end{cases}$$

Here cohomology means sheaf cohomology with coefficients in the constant sheaf $\underline{A}$ on X (which does not agree with singular cohomology in this generality).

Proof. If X is not connected, then there exists a map $f : X \to S^0$ which is not homotopic to a constant map. This proves (1). To prove (2), suppose that $\mathcal{F}$ Is a locally constant sheaf of sets on X. Pick a point $x \in X$ and set $S = \mathcal{F}_x$. Since X is connected, every stalk of $\mathcal{F}$ is (noncanonically) isomorphic to S. Let G denote the permutation group of S (regarded as a discrete group) and let BG denote its classifying space. Since X is paracompact, isomorphism classes of locally constant sheaves of sets on X with stalks isomorphic to S are in bijection with homotopy classes of maps $f : X \to \mathrm{BG}$. Since any such map $f : X \to \mathrm{BG}$ is homotopic to a constant map, any such locally constant sheaf is actually constant.

To prove (3), we note that since X is paracompact, we can identify $\mathrm{H}^n(X; \underline{A})$ with the set of homotopy classes of maps from X into the Eilenberg-MacLane space $K(A, n)$. Since X has trivial shape, this can be identified with the set of connected components $\pi_0 K(A, n) \simeq \begin{cases} A & \mathrm{if} * = 0 \\ 0 & \text{otherwise.} \end{cases}$. □

Remark 18. Definition 15 is perhaps only appropriate when the topological space X is paracompact; for many purposes, it is the conclusions of Proposition 17 which are important. For us, this will be irrelevant: we will only be interested in the case where X is a compact Hausdorff space.

Definition 19. Let $f : X \to Y$ be a map of Hausdorff spaces. We will say that f is *cell-like* if it satisfies the following conditions:

(1) The map f is closed.

(2) For each point $y \in Y$, the fiber $X_y = f^{-1}\{y\}$ is compact.

(3) For each point $y \in Y$, the fiber X_y has trivial shape.

Remark 20. Conditions (1) and (2) of Definition 19 say that the map $f : X \to Y$ is *proper*.

Warning 21. Definition 19 is generally only behaves well if we make some additional assumptions on the spaces X and Y involved: for example, if we assume they are absolute neighborhood retracts or that they are paracompact and finite-dimensional. See Remark **??** below.

Example 22. Any homeomorphism of Hausdorff spaces is a cell-like map.

Proposition 23. *Let f be a cell-like map of CW complexes. Then f is a homotopy equivalence.*

Proof. We first argue that f is an equivalence on fundamental groupoids. For this, it suffices to show that the pullback functor f^* induces an equivalence from the category of locally constant sheaves (of sets) on Y to the category of locally constant sheaves (of sets) on X. This amounts to two assertions:

(a) For every locally constant sheaf $\mathcal{F}$ on Y, the unit map

$$\mathcal{F} \to f_* f^* \mathcal{F}$$

is an isomorphism.

(b) For every locally constant sheaf $\mathcal{G}$ on X, the pushforward $f_* \mathcal{G}$ is locally constant and the counit map $f^* f_* \mathcal{G} \to \mathcal{G}$ is an isomorphism.

Note that if $\mathcal{G}$ is a locally constant sheaf on X. Fix a point $x \in X$, let $S = \mathcal{G}_x$ be the stalk of $\mathcal{G}$ at the point x, and let $y = f(x)$. Since the fiber X_y is connected, we can choose an open neighborhood V of X_y such that the stalk $\mathcal{G}_{x'}$ is (noncanonically) isomorphic to S for each $x' \in U$. As in the proof of Proposition 17, this implies that $\mathcal{G}|_U$ is classified by a map $U \to \mathrm{BG}$, where G is the group of permutations of S. This map is nullhomotopic on the fiber X_y and therefore on some neighborhood of X_y. Since f is proper, we may assume without loss of generality that such a neighborhood has the form $f^{-1}V$ for some open set $V \subseteq Y$ containing the point y. It follows that $\mathcal{G}|_{f^{-1}(V)}$ is isomorphic to the constant sheaf $\underline{S}$ with the value S. Since f is a proper map, the stalk of $f_* \mathcal{G}$ at any point $y' \in V$ can be identified with the set of continuous maps from $X_{y'}$ into S (where S has the discrete topology). Since each $X_{y'}$ s connected, it follows that $(f_* \mathcal{G})|_V$ can also be identified with the constant sheaf having the value S. This proves (b). Using the properness of f, we see that assertion (a') reduces to the following:

(a') For every set S, the canonical map $S \to \mathrm{Hom}(X_y, S)$ is a bijection.

This follows immediately from the connectedness of X_y.

To complete the proof that f is a homotopy equivalence, it will suffice to show that for every local system of abelian groups $\mathcal{A}$ on Y, the canonical map

$$\theta : \mathrm{H}^*(Y; \mathcal{A}) \to \mathrm{H}^*(X; f^* \mathcal{A})$$

is an isomorphism. Since X and Y are CW complexes, we can regard $\mathcal{A}$ as a locally constant sheaf on Y and identify the domain and codomain of θ with *sheaf* cohomology. To prove this, it suffices to prove the stronger *local* assertion that the unit map $\mathcal{A} \to Rf_* f^* \mathcal{A}$ is an isomorphism in the derived category of sheaves on Y. Since f is proper, the cohomologies of the stalks of $Rf_* f^* \mathcal{A}$ are given by the sheaf cohomology groups $\mathrm{H}^*(X_y; \mathcal{A}_y)$, so that the desired result follows from Proposition 17. □

Proposition 23 does not need the full strength of the assumption that X and Y are CW complexes: it is enough to know that X and Y have the homotopy type of CW complexes (so that homotopy equivalences are detected by Whitehead's theorem), that local systems can be identified with locally constant sheaves, and that singular cohomology agrees with sheaf cohomology. Consequently, it remains valid if we assume only that X and Y are open subsets of CW complexes or absolute neighborhood retracts (ANRs).

Corollary 24. *Let $f : X \to Y$ be a proper map of CW complexes (or ANRs). Then f is cell-like if and only if, for every open set $U \subseteq Y$, the induced map $f^{-1}U \to U$ is a homotopy equivalence.*

Proof. The "only if" direction follows from Proposition 23. Conversely, suppose that for each $U \subseteq Y$ the map $f^{-1}U \to U$ is a homotopy equivalence. Pick a point $y \in Y$, a CW complex Z, and a map $g : X_y \to Z$; we wish to prove that g is nullhomotopic. Then g factors through a finite subcomplex of Z; we may therefore assume without loss of generality that Z is finite. We can extend g to a continuous map $\overline{g} : V \to Z$ for some open set $V \subseteq X$ containing X_y (this follows from the fact that the finite CW complex Z is an absolute neighborhood retract). Because f is proper, V contains a set of the form $f^{-1}(U)$ where U is a neighborhood of y in Y. Using the local contractibility of Y, we may further assume that U is contractible. Then $f^{-1}(U)$ is also contractible. The map g factors as a composition

$$X_y \hookrightarrow f^{-1}U \stackrel{\overline{g}}{\to} Z,$$

and is therefore nullhomotopic. □

We will prove the following result in the next lecture:

Proposition 25. *Let $f : X \to Y$ be a map of finite CW complexes. Assume that f is cell-like, cellular, and that for every cell $e \subseteq Y$, the inverse image $f^{-1}e$ is a union of cells. Then f is a simple homotopy equivalence.*

Remark 26. In the statement of Proposition 25, the second and third hypothesis on f are not necessary. However, the proof requires infinite-dimensional methods.