A PullBack Square (Lecture 32)

November 24, 2014

Let X be a simplicial set. As before, we let $\mathcal{C}_X$ denote the category whose objects are diagrams

$$\begin{array}{ccc} & Y & \\ \nearrow & & \searrow \\ X & \xrightarrow{\mathrm{id}} & X \end{array}$$

where Y is obtained from X by adding finitely many simplices. Let s denote the collection of cell-like maps in $\mathcal{C}_X$, let h denote the collection of weak homotopy equivalences in $\mathcal{C}_X$, and let $\mathcal{C}_X^h$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects where the map $X \to Y$ is a weak homotopy equivalence. Our goal in this lecture (and the next) is to complete the second part of this course by establishing the following result:

Proposition 1. *The diagram*

$$\begin{array}{ccc} K(\mathcal{C}_X^h, s) & \longrightarrow & K(\mathcal{C}_X^h, h) \\ \downarrow & & \downarrow \\ K(\mathcal{C}_X, s) & \longrightarrow & K(\mathcal{C}_X, h) \end{array}$$

is a homotopy pullback square.

We will prove Proposition 1 by analyzing the K-theory space $K(\mathcal{C}_X, h)$ (which we know to be homotopy equivalent to $\Omega^\infty A^{\mathrm{free}}(X)$) and eventually showing that it can be identified with the homotopy quotient of $K(\mathcal{C}_X, s)$ by the action of $K(\mathcal{C}_X^h, s)$.

As a first step, it will be convenient to replace $\mathcal{C}_X$ by something slightly closer to $\mathcal{C}_X^h$.

Definition 2. For each integer n, let $\mathcal{C}_X^{(n)}$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects for which the map $X \to Y$ is n-connected.

Lemma 3. *For each integer n, the inclusion $\mathcal{C}_X^{(n)} \hookrightarrow \mathcal{C}_X$ induces homotopy equivalences*

$$K(\mathcal{C}_X^{(n)}, s) \to K(\mathcal{C}_X, s) \qquad K(\mathcal{C}_X^{(n)}, h) \to K(\mathcal{C}_X, h).$$

Proof. We will give the proof of the second assertion; the proof of the first is similar. When $n = -1$, there is nothing to prove. Proceeding by induction on n, we are reduced to proving that each of the inclusions $\mathcal{C}_X^{(n+1)} \hookrightarrow \mathcal{C}_X^{(n)}$ inducse a homotopy equivalence $K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h)$. Let Y be an object of $\mathcal{C}_X$, given by a diagram

$$\begin{array}{ccc} & Y & \\ \nearrow & & \searrow^{r} \\ X & \xrightarrow{\mathrm{id}} & X. \end{array}$$

Let $M(r) = (Y \times \Delta^1) \amalg_{Y \times \{1\}} X$ denote the mapping cylinder of r and let $F(Y) = X \amalg_Y M(r)$ denote the two-sided mapping cylinder of r. The construction $Y \mapsto F(Y)$ induces a functor from $\mathcal{C}_X$ to itself which carries $\mathcal{C}_X^{(n)}$ into $\mathcal{C}_X^{(n+1)}$; in particular, it carries both $\mathcal{C}_X^{(n)}$ and $\mathcal{C}_X^{(n+1)}$ to themselves. Note that F preserves cofibrations, pushouts, weak homotopy equivalences, and cell-like maps. It therefore induces maps on K-theory. Applying the two-out-of-six property to the diagram of spaces

$$K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h) \xrightarrow{F} K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h),$$

we are reduced to showing that F induces homotopy equivalences from $K(\mathcal{C}_X^{(n)}, h)$ and $K(\mathcal{C}_X^{(n+1)}, h)$ to themselves. In fact, we claim that on both K-theory spaces F acts by (-1): this follows by applying the additivity theorem to the natural cofiber sequence

$$Y \to M(r) \to F(Y),$$

since the functor $Y \mapsto M(r)$ is related by a cell-like natural transformation to the constrant functor $Y \mapsto X$. □

By virtue of Lemma 3, it will suffice to show that the diagram

$$\begin{array}{ccc} K(\mathcal{C}_X^h, s) & \longrightarrow & K(\mathcal{C}_X^h, h) \\ \downarrow & & \downarrow \\ K(\mathcal{C}_X^{(1)}, s) & \longrightarrow & K(\mathcal{C}_X^{(1)}, h) \end{array}$$

is a homotopy pullback square.

Note that $K(\mathcal{C}_X^{(1)}, h)$ can be obtained as the geometric realization of the simplicial object of Set_Δ given by

$$[n] \mapsto \mathrm{N}(hS_n\,\mathcal{C}_X^{(1)}).$$

Let us fix n for the moment, and consider the category $hS_n\,\mathcal{C}_X^{(1)}$: the objects of this category can be identified with diagrams

$$X \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_n \to X$$

where all but the last map are 1-connected cofibrations (each adding finitely many simplices) and the composition is the identity, and the morphisms are levelwise weak homotopy equivalences. Let us denote such an object simply by $\vec{Y}$. We would like to analyze $hS_n\,\mathcal{C}_X^{(1)}$ in terms of the subcategory where the morphisms are levelwise cell-like maps. To this end, let us consider a bisimplicial set $\mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet}$ whose (p,q)-simplices are diagrams

$$\begin{array}{ccccc} \vec{Y}_{0,0} & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_{0,q} \\ \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \vec{Y}_{p,0} & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_{p,q} \end{array}$$

where the horizontal maps are levelwise weak homotopy equivalences and the vertical maps are levelwise cell-like.

Lemma 4 (Swallowing Lemma)**.** *In the situation above, the canonical map*

$$\mathrm{N}(hS_n\,\mathcal{C}_X^{(1)})_\bullet \simeq \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{0,\bullet} \to \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet}$$

is a homotopy equivalence (after geometric realization).

Proof. It will suffice to show that for each $p \geq 0$, the natural map $\mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{0,\bullet} \to \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet}$ is a weak homotopy equivalence of simplicial sets. Note that the target can be identified with the nerve of the category $\mathcal{E}$ whose objects are diagrams

$$\vec{Y}_0 \to \vec{Y}_1 \to \cdots \to \vec{Y}_p$$

of (levelwise) cell-like maps in $hS_n\,\mathcal{C}_X^{(1)}$. The diagonal map $hS_n\,\mathcal{C}_X^{(1)} \hookrightarrow \mathcal{E}$ admits a left inverse, given by the construction

$$\vec{Y}_0 \to \vec{Y}_1 \to \cdots \to \vec{Y}_p \mapsto \vec{Y}_0.$$

This left inverse is also a right homotopy inverse by means of the evident natural map

$$\begin{array}{ccccccc}
\vec{Y}_0 & \xrightarrow{\mathrm{id}} & \vec{Y}_0 & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vec{Y}_0 & \longrightarrow & \vec{Y}_1 & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_p.
\end{array}$$

□

It will be convenient to consider a slightly smaller bisimplicial set. We say that a morphism $\vec{Y} \to \vec{Y}'$ in $hS_n\,\mathcal{C}_X^{(1)}$ is a *cofibration* if the induced map $Y_i' \amalg_{Y_i} Y_{i+1} \to Y_{i+1}'$ is a monomorphism of simplicial sets for each i. Let $\mathrm{N}''(h\,\mathcal{C}_X^{(1)})_{\bullet\bullet}$ denote the bisimplicial set whose objects are diagrams

$$\begin{array}{ccccc}
\vec{Y}_{0,0} & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_{0,q} \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\vec{Y}_{p,0} & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_{p,q}
\end{array}$$

where the horizontal maps are cofibrations and levelwise weak homotopy equivalences and the vertical maps are cell-like.

Lemma 5. *The inclusion of bisimplicial sets*

$$\mathrm{N}''(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet} \hookrightarrow \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{\bullet,\bullet}$$

is a weak homotopy equivalence (after geometric realization).

Proof. It will suffice to show that for each integer $p \geq 0$, the inclusion

$$\mathrm{N}''(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet} \hookrightarrow \mathrm{N}'(hS_n\,\mathcal{C}_X^{(1)})_{p,\bullet}$$

is a weak homotopy equivalence. In other words, if we let $\mathcal{E}$ be the category appearing in the proof of Lemma 4 and we let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the subcategory of $\mathcal{E}$ whose morphisms are given by diagrams

$$\begin{array}{ccccccc}
\vec{Y}_0 & \xrightarrow{\mathrm{id}} & \vec{Y}_1 & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vec{Y}_0' & \longrightarrow & \vec{Y}_1' & \longrightarrow & \cdots & \longrightarrow & \vec{Y}_p',
\end{array}$$

where the vertical maps are cofibrations (as well as being weak homotopy equivalences), then we wish to show that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence. Let us assume for simplicity that $p = n = 0$ (the proof in the general case is differs only by notation): then $\mathcal{E}$ is the subcategory of $\mathcal{C}_X^{(1)}$ whose morphisms are weak homotopy equivalences, and $\mathcal{E}_0$ is the subcategory of $\mathcal{C}_X^{(1)}$ whose morphisms are trivial cofibrations. We will prove that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence by showing that it is right cofinal. To this end, fix an object $Y \in \mathcal{E}$; we wish to show that the category $\mathcal{D} = \mathcal{E}_0 \times_{\mathcal{E}} \mathcal{E}_{/Y}$ is weakly contractible. Unwinding the definitions, we can identify the objects of $\mathcal{D}$ with weak homotopy equivalences $f : Y' \to Y$ in $\mathcal{C}_X^{(1)}$. To prove that $\mathcal{D}$ is weakly contractible, it suffices to observe that every such object is connected to the identity map $\mathrm{id} : Y \to Y$ by a canonical zig-zag of trivial cofibrations

$$Y' \hookrightarrow (M(f) \amalg_{X \times \Delta^1} X) \hookleftarrow Y$$

where $M(f) = (Y' \times \Delta^1) \amalg_{Y' \times \{1\}} Y$ is the mapping cylinder of f. □

Let us now reorganize a bit. For each $q \geq 0$, let $F_q(\mathcal{C}_X^{(1)})$ denote the category whose objects are sequences of trivial cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}_X^{(1)}$. Then we can regard $F_q\,\mathcal{C}_X^{(1)}$ as a category with cofibrations (defined as above, with the roles of n and q switched) and weak equivalences (given by the collection s of levelwise cell-like maps). This category with cofibrations and weak equivalences depends functorially on $[q]$, so we can regard $F_\bullet\,\mathcal{C}_X^{(1)}$ as a *simplicial* category with cofibrations and weak equivalences. Unwinding the definitions, we have

$$K(F_q\,\mathcal{C}_X^{(1)}) \simeq |\,\mathrm{N}(hS_\bullet\,\mathcal{C}_X^{(1)})_{\bullet,q}|.$$

Passing to the geometric realization as $[q]$ varies and invoking Lemmas 4 and 5, we obtain a homotopy equivalence

$$K(\mathcal{C}_X^{(1)}, h) \simeq |K(F_\bullet\,\mathcal{C}_X^{(1)}, s)|.$$

Given a cofibration $Y \hookrightarrow Y'$ in $\mathcal{C}_X$, let Y'/Y denote the pushout $Y' \amalg_Y X$. It is clear that if $Y \hookrightarrow Y'$ is a weak homotopy equivalence, then the quotient Y'/Y is weakly homotopy equivalent to X. If Y' and Y both belong to $\mathcal{C}_X^{(1)}$, then the converse holds: this follows from the observation that for any local system of abelian groups $\mathcal{A}$ on X, we have an isomorphism

$$\mathrm{H}_*(Y', Y; \mathcal{A}\,|_{Y'}) \simeq \mathrm{H}_*(Y'/Y, X; \mathcal{A}\,|_{Y'/Y}).$$

It follows that $F_q(\mathcal{C}_X^{(1)})$ admits an alternative description: it can be identified with the category whose objects are sequences of cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}_X^{(1)}$ where each quotient Y_i/Y_{i-1} belongs to $\mathcal{C}_X^h$.

There is a natural map

$$\theta_q : \mathcal{C}_X^{(1)} \times (\mathcal{C}_X^h)^q \to F_q(\mathcal{C}_X^{(1)}),$$

given on objects by

$$(Y, (Z_1, \ldots, Z_q)) \mapsto (Y \hookrightarrow Y \amalg_X Z_1 \hookrightarrow \cdots \hookrightarrow Y \amalg_X Z_1 \amalg_X \cdots \amalg_X Z_q)$$

This induces a map on K-theory spaces

$$K(\mathcal{C}_X^{(1)}, s) \times K(\mathcal{C}_X^h, s)^q \to K(F_q(\mathcal{C}_X^{(1)}), s).$$

Passing to the geometric realization as q varies, we obtain a map

$$K(\mathcal{C}_X^{(1)}, s)/K(\mathcal{C}_X^h, s)^q \to |K(F_\bullet\,\mathcal{C}_X^{(1)}, s)| \simeq K(\mathcal{C}_X^{(1)}, h).$$

To prove Proposition 1, it will suffice to show that this map is a homotopy equivalence. In fact, we will prove something stronger: each of the maps θ_q induces a homotopy equivalence at the level of K-theory. Note that θ_q has a left homotopy inverse ρ, given by the construction

$$(Y_0 \hookrightarrow \cdots \hookrightarrow Y_q) \mapsto (Y_0, (Y_1/Y_0, \cdots, Y_q/Y_{q-1})).$$

The composition $\theta_q \circ \rho$ is not homotopic to the identity at the level of categories, but induces the identity map on K-theory spaces (up to homotopy) by virtue of the additivity theorem.

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*