The Whitehead Space II (Lecture 31)

November 19, 2014

Let X be a simplicial set. As in the previous lecture, we let $\mathcal{D}_X$ denote the subcategory of $(\mathrm{Set}_\Delta)_{X/}$ spanned by those objects $i : X \hookrightarrow Y$ which are trivial cofibrations of simplicial sets obtained by adding finitely many simplices to X, and whose morphisms are cell-like maps. We let $W(X)$ denote the nerve of $\mathcal{D}_X$. Our goal in this lecture is to show that if X is finite, then $W(X)$ can be identified with the homotopy fiber product $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$. Our first step is to establish the following result (already used without proof in the previous lecture):

Proposition 1. *The functor $X \mapsto W(X)$ preserves weak homotopy equivalences.*

We will deduce Proposition 1 from two special cases:

Lemma 2. *Let X be a finite simplicial set. Then the "last vertex" map $\mathrm{Sd}(X) \to X$ induces a weak homotopy equivalence $W(\mathrm{Sd}(X)) \to W(X)$.*

Lemma 3. *Let X be a finite simplicial set. Then the projection map $X \times \Delta^1 \to X$ induces a weak homotopy equivalence $W(X \times \Delta^1) \to W(X)$.*

Proof of Proposition 1. We first show that if $f : X \to Y$ is a weak homotopy equivalence of finite simplicial sets, then the induced map $W(X) \to W(Y)$ is a weak homotopy equivalence. Since X is not a Kan complex, the map f need not be a homotopy equivalence. However, there exists a homotopy inverse to f after fibrant replacement: that is, a map $g : Y \to \mathrm{Ex}^\infty X$ such that the unit map $X \to \mathrm{Ex}^\infty X$ is homotopic to $g \circ f$, and $\mathrm{Ex}^\infty(f) \circ g$ is homotopic to the unit map $Y \to \mathrm{Ex}^\infty Y$. Since X and Y are finite, we can replace Ex^∞ by Ex^n for $n \gg 0$. In this case, we can identify g with a map $G : \mathrm{Sd}^n(Y) \to X$, and we have homotopies

$$h : \mathrm{Sd}^n(X \times \Delta^1) \to X$$

$$h' : \mathrm{Sd}^n(Y \times \Delta^1) \to Y.$$

To show that these maps yield homotopies after applying W, it suffices to show that the maps

$$W(\mathrm{Sd}^n(X \times \Delta^1)) \to W(X)$$

$$W(\mathrm{Sd}^n(X)) \to W(X)$$

are weak homotopy equivalences, and similarly for Y; these assertions are immediate consequences of Lemmas 2 and 3.

It follows from the above argument that when restricted to finite simplicial sets, the functor $W : \mathrm{Set}_\Delta \to \mathcal{S}$ preserves weak homotopy equivalences, and therefore induces a functor of ∞-categories $u : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$. The functor u admits an essentially extension $U : \mathcal{S} \to \mathcal{S}$ which commutes with filtered colimits. Since W commutes with filtered colimits, it follows that it is given by the composition

$$\mathrm{Set}_\Delta \to \mathcal{S} \xrightarrow{U} \mathcal{S}.$$

$\square$

Proof of Lemma 3. Pushout along the projection map $X \times \Delta^1 \to X$ induces a functor $f : \mathcal{D}_{X \times \Delta^1} \to \mathcal{D}_X$. Consider the functor $g : \mathcal{D}_X \to \mathcal{D}_{X \times \Delta^1}$ given by $Y \mapsto Y \times \Delta^1$. We claim that, after passing to nerves, these maps are mutually inverse homotopy equivalences relating $W(X \times \Delta^1)$ and $W(X)$. Note that $f \circ g : \mathcal{D}_X \to \mathcal{D}_X$ is the functor given by

$$Y \mapsto X \amalg_{X \times \Delta^1} (Y \times \Delta^1).$$

At the level of nerves, this is homotopic to the identity map, since the projection $Y \times \Delta^1 \to Y$ induces a cell-like map

$$X \amalg_{X \times \Delta^1} (Y \times \Delta^1) \to Y.$$

The functor $g \circ f : \mathcal{D}_{X \times \Delta^1} \to \mathcal{D}_{X \times \Delta^1}$ is given by

$$Y \mapsto (Y \amalg_{X \times \Delta^1} X) \times \Delta^1.$$

In this case, we have a two-step homotopy to the identity, given by the diagram

$$(Y \amalg_{X \times \Delta^1} X) \times \Delta^1 \leftarrow Y \times \Delta^1 \to Y.$$

□

Proof of Lemma 2. We wish to show that the functor

$$f : \mathcal{D}_{\mathrm{Sd}(X)} \to \mathcal{D}_X$$

$$Y \mapsto Y \amalg_{\mathrm{Sd}(X)} X$$

induces a weak homotopy equivalence on nerves. We will show that the construction

$$g : \mathcal{D}_X \to \mathcal{D}_{\mathrm{Sd}(X)}$$

$$Y \mapsto \mathrm{Sd}(Y)$$

provides a homotopy inverse. Note that the composite map $f \circ g : \mathcal{D}_X \to \mathcal{D}_X$ is related to the identity functor by a cell-like natural transformation

$$\mathrm{Sd}(Y) \amalg_{\mathrm{Sd}(X)} X \to Y.$$

The other direction is a bit trickier: the composite functor $g \circ f : \mathcal{D}_{\mathrm{Sd}(X)} \to \mathcal{D}_{\mathrm{Sd}(X)}$ carries an object $Y \in \mathcal{D}_{\mathrm{Sd}(X)}$ to the object $\mathrm{Sd}(Y \amalg_{\mathrm{Sd}(X)} X) = \mathrm{Sd}(Y) \amalg_{\mathrm{Sd}^2(X)} \mathrm{Sd}(X)$. In other words, we can identify $(g \circ f)(Y)$ with the image of $\mathrm{Sd}(Y) \in \mathcal{D}_{\mathrm{Sd}^2(X)}$ under the functor $\mathcal{D}_{\mathrm{Sd}^2(X)} \to \mathcal{D}_{\mathrm{Sd}(X)}$ which is obtained from the map $\mathrm{Sd}(e) : \mathrm{Sd}^2(X) \to \mathrm{Sd}(X)$, where $e : \mathrm{Sd}(X) \to X$ is the "last vertex map". Note that e does not coincide with the "last vertex" map $\mathrm{Sd}^2(X) \to \mathrm{Sd}(X)$, but it is simplicially homotopic to it, and therefore (by virtue of Lemma 3) induces a homotopic map from $W(\mathrm{Sd}^2(X))$ to $W(\mathrm{Sd}(X))$. We are therefore reduced to proving that the functor $Y \mapsto \mathrm{Sd}(Y) \amalg_{\mathrm{Sd}^2(X)} \mathrm{Sd}(X)$ is homotopic to the identity, where $\mathrm{Sd}^2(X)$ maps to $\mathrm{Sd}(X)$ via the "last vertex" map. This follows from the first part of the proof (applied to $\mathrm{Sd}(X)$ rather than X). □

It will be useful for us to consider a slight variant of the category $\mathcal{D}_X$. From this point forward, let us assume that the simplicial set X is finite. Let $\mathcal{D}_X^+$ denote the subcategory of $(\mathrm{Set}_\Delta)_{X/}$ whose objects are weak homotopy equivalences $X \to Y$ of finite simplicial sets, and whose morphisms are cell-like maps. Then $\mathcal{D}_X^+$ contains $\mathcal{D}_X$ as a full subcategory: the only difference is that we no longer require the structure map $X \to Y$ to be a cofibration.

Proposition 4. *For every finite simplicial set X, the inclusion $\mathcal{D}_X \hookrightarrow \mathcal{D}_X^+$ induces a weak homotopy equivalence of nerves.*

Proof. For each morphism $f : X \to Y$, let $M(f) = (X \times \Delta^1) \amalg_{X \times \{1\}} Y$ denote the mapping cylinder of f. Then the construction

$$(f : X \to Y) \mapsto (X \times \{0\} \hookrightarrow M(f))$$

determines a functor from $\mathcal{D}_X^+$ into $\mathcal{D}_X$. Using the natural cell-like map $M(f) \to Y$, we see that this functor determines a deformation retraction of $\mathrm{N}(\mathcal{D}_X^+)$ into $\mathrm{N}(\mathcal{D}_X)$. □

Note that the enlargement $\mathcal{D}_X \mapsto \mathcal{D}_X^+$ comes at a price: if $X \to X'$ is a map of finite simplicial sets, the construction

$$Y \mapsto Y \amalg_X X'$$

generally does not preserve cell-like maps (or weak homotopy equivalences), and therefore does not induce a functor from $\mathcal{D}_X^+$ to $\mathcal{D}_{X'}^+$. However, we get a different sort of functoriality as compensation: if $f : X \to X'$ is a weak homotopy equivalence, then composition with f induces a map $\mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$. We will need the following variant of Proposition 1:

Proposition 5. *Let $f : X \to X'$ be a weak homotopy equivalence of finite simplicial sets. Then composition with f induces a weak homotopy equivalence $\mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$.*

Proof. Arguing as in the proof of Proposition 1, it suffices to treat the case of the maps

$$\mathrm{Sd}(X) \to X \qquad X \times \Delta^1 \to X.$$

Using Propositions 1 and 4, we are reduced to proving the the composite functor

$$\mathcal{D}_X \to \mathcal{D}_{X'} \hookrightarrow \mathcal{D}_{X'}^+ \to \mathcal{D}_X^+$$

$$Y \mapsto Y \amalg_X X'$$

is a weak homotopy equivalence. Since f is cell-like, this functor is related to the inclusion $\mathcal{D}_X \hookrightarrow \mathcal{D}_X^+$ by a natural transformation $Y \to Y \amalg_X X'$; the desired result now follows from Proposition 4. □

Fix a finite simplicial set X. For each $n \geq 0$, the construction $[m] \mapsto \mathrm{Sd}^n(X \times \Delta^m)$ determines a cosimplicial object of $\mathcal{C}$. We therefore obtain a simplicial category

$$\mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^\bullet)}.$$

After taking nerves, we obtain a simplicial space which is equivalent to the constant simplicial space with the value

$$\mathrm{N}(\mathcal{D}^+_{\mathrm{Sd}^n(X)}) \simeq \mathrm{N}(\mathcal{D}_X^+) \simeq \mathrm{N}(\mathcal{D}_X) \simeq W(X).$$

We may therefore identify $W(X)$ with the geometric realization

$$|\varinjlim_n \mathrm{N}\,\mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^\bullet)}|.$$

Let $\mathcal{E}$ denote the category whose objects are finite simplicial sets Y and whose morphisms are cell-like maps. Then each of the categories $\mathcal{D}^+_{\mathrm{Sd}^n(X \times \Delta^m)}$ is cofibered in sets over $\mathcal{E}$, and can therefore be identified with the Grothendieck construction on the functor

$$f_{n,m} : \mathcal{E} \to \mathcal{S}\mathrm{et}$$

which assigns to each object $Y \in \mathcal{E}$ the set of all weak homotopy equivalences

$$\mathrm{Sd}^n(X \times \Delta^m) \to Y.$$

It follows that the nerve of $\mathcal{D}^+_{\mathrm{Sd}^n(X\times\Delta^m)}$ can be identified with the homotopy colimit of the diagram $f_{n,m}$. It follows that

$$\varinjlim_n \mathrm{N}\,\mathcal{D}^+_{\mathrm{Sd}^n(X\times\Delta^\bullet)}$$

can be identified with the homotopy colimit of the functor

$$f_m : \mathcal{E} \to \mathcal{S}\mathrm{et}$$

$$f_m(Y) = \mathrm{Hom}'(X\times\Delta^m, \mathrm{Ex}^\infty Y).$$

where $\mathrm{Hom}'(X\times\Delta^m, \mathrm{Ex}^\infty Y)$ is the subset of $\mathrm{Hom}(X\times\Delta^m, \mathrm{Ex}^\infty Y)$ consisting of weak homotopy equivalences. Passing to the geometric realization, we can identify $W(X)$ with the homotopy colimit of the diagram

$$\mathcal{E} \to \mathrm{Set}_\Delta$$

$$Y \mapsto H(X, \mathrm{Ex}^\infty Y)$$

where $H(X, \mathrm{Ex}^\infty Y)$ is the simplicial set parametrizing homotopy equivalences from X to $\mathrm{Ex}^\infty Y$. We saw in Lecture 12 that we can identify $\mathcal{M}$ with the nerve of $\mathcal{E}$; this identification induces an equivalence

$$W(X) \simeq \varinjlim_{Y\in\mathcal{E}} H(X, \mathrm{Ex}^\infty Y) \simeq \mathrm{N}(\mathcal{E}) \times_{\mathcal{M}^h} \{X\} \simeq \mathcal{M}\times_{\mathcal{M}^h}\{X\}.$$

We conclude by discussing the extent to which the homotopy equivalence $W(X) \simeq \mathcal{M}\times_{\mathcal{M}^h}\{X\}$ can be made functorial in X. By virtue of the above discussion, this amounts to the question of how functorially we can identify the spaces $\mathrm{N}(\mathcal{D}_X)$ with $\mathrm{N}(\mathcal{D}^+_X)$. It follows from Propositions 1 and 5 that the constructions

$$X \mapsto \mathrm{N}(\mathcal{D}_X) \qquad X \mapsto \mathrm{N}(\mathcal{D}^+_X)$$

define functors

$$u : \mathcal{E} \to \mathcal{S}^{\simeq} \qquad v : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}^{\simeq}.$$

We will prove the following:

Proposition 6. *The functor u and v are homotopic to one another (after identifying $\mathcal{S}^{\simeq}$ with its opposite).*

Warning 7. We can extend u and v to functors defined on the larger category whose objects are finite simplicial sets and whose morphisms are weak homotopy equivalences. However, these enlargements are *not* equivalent to one another (note that if they were, then the fibration $\mathcal{M} \to \mathcal{M}^h$ would be classified by the functor $X \mapsto W(X)$, and would therefore admit a section).

To prove Proposition 6, we begin by applying the Grothendieck construction to the assignments

$$X \mapsto \mathcal{D}_X \qquad X \mapsto \mathcal{D}^+_X$$

to produce coCartesian fibrations

$$\mathcal{D} \to \mathcal{E} \qquad \mathcal{D}^+ \to \mathcal{E}^{\mathrm{op}}:$$

the objects of $\mathcal{D}$ are trivial cofibrations $i : X \to Y$ of finite simplicial sets, and the objects of $\mathcal{D}^+$ are weak homotopy equivalences $i : X \to Y$ of finite simplicial sets. Morphisms in $\mathcal{D}$ are given by commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

where the vertical maps are cell-like and the horizontal maps are trivial cofibrations, and morphisms in $\mathcal{D}^+$ are given by commutative diagrams

$$\begin{CD} X @>>> Y \\ @AAA @VVV \\ X' @>>> Y' \end{CD}$$

where the vertical maps are cell-like and the horizontal maps are weak homotopy equivalences.

Remark 8. There is a bit of work hidden in this description of $\mathcal{D}$. *A priori*, the morphisms in the relevant Grothendieck construction are given by commutative diagrams

$$\begin{CD} X @>>> Y \\ @VfVV @VgVV \\ X' @>>> Y' \end{CD}$$

where the horizontal maps are trivial cofibrations, f is cell-like, and the induced map $g' : Y \amalg_X X' \to Y'$ is cell-like. But the assumption that f is cell-like guarantees that the map $Y \to Y \amalg_X X'$ is cell-like, from which it follows that g is cell-like if and only if g' is cell-like.

These coCartesian fibrations induce maps of spaces

$$U : \mathrm{N}(\mathcal{D}) \to \mathrm{N}(\mathcal{E}) \qquad V : \mathrm{N}(\mathcal{D}^+) \to \mathrm{N}(\mathcal{E}^{\mathrm{op}}).$$

Using Propositions 1 and 5 and Quillen's Theorem B, we see that the homotopy fibers of these maps (over an object $X \in \mathcal{E}$) can be identified with $\mathrm{N}(\mathcal{D}_X)$ and $\mathrm{N}(\mathcal{D}^+_X)$, respectively. Consequently, Proposition 6 can be reformulated as follows: the natural homotopy equivalence $\mathrm{N}(\mathcal{E}) \simeq \mathrm{N}(\mathcal{E}^{\mathrm{op}})$ can be lifted to an equivalence between U and V (regarded as objects in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$ of morphisms in the ∞-category of spaces).

To prove this, let $\mathrm{TwArr}(\mathcal{E})$ denote the "twisted arrow category" of $\mathcal{E}$: that is, the category whose objects are weak homotopy equivalences $f : X_0 \to X_1$ of finite simplicial sets, where a morphism from $f : X_0 \to X_1$ to $f' : X'_0 \to X'_1$ is a commutative diagram

$$\begin{CD} X_0 @>f>> X_1 \\ @AAA @VVV \\ X'_0 @>f'>> X'_1. \end{CD}$$

$(f : X_0 \to X_1) \to (X_0, X_1)$ determines a coCartesian fibration

$$\mathrm{TwArr}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times \mathcal{E}\,.$$

In particular, we have coCartesian fibrations

$$\mathcal{E}^{\mathrm{op}} \xleftarrow{e_0} \mathrm{TwArr}(\mathcal{E}) \xrightarrow{e_1} \mathcal{E}$$

The fibers of these coCartesian fibrations are weakly contractible (since they have initial objects), so Quillen's Theorem B implies that e_0 and e_1 are weak homotopy equivalences; the diagram of spaces

$$\mathrm{N}(\mathcal{E}^{\mathrm{op}}) \leftarrow \mathrm{N}(\mathrm{TwArr}(\mathcal{E})) \to \mathrm{N}(\mathcal{E})$$

supplies a concrete combinatorial description of the natural equivalence between $\mathrm{N}(\mathcal{E}^{\mathrm{op}})$ and $\mathrm{N}(\mathcal{E})$ (in the ∞-category of spaces). We may therefore reformulate Proposition 6 as follows: the spaces $\mathrm{N}(\mathcal{D}^+ \times_{\mathcal{E}^{\mathrm{op}}} \mathrm{TwArr}(\mathcal{E}))$

and $N(\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E}))$ are equivalent (in the ∞-category of spaces over $\mathrm{TwArr}(\mathcal{E})$). Note that we can identify the objects of $\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E})$ with diagrams of finite simplicial sets $X_0 \xrightarrow{f} X_1 \xhookrightarrow{g} Y$ where f is a weak homotopy equivalence and g is a trivial cofibration, and we can identify the objects of $\mathcal{D}^+ \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E})$ with diagrams

$$Y \xleftarrow{h} X_0 \xrightarrow{f} X_1$$

where f and h are weak homotopy equivalences. The construction $(f, g) \mapsto (f, g \circ f)$ determines a functor

$$\mathcal{D} \times_{\mathcal{E}} \mathrm{TwArr}(\mathcal{E}) \to \mathcal{D}^+ \times_{\mathcal{E}^{\mathrm{op}}} \mathrm{TwArr}(\mathcal{E})$$

compatible with the projection to $\mathrm{TwArr}(\mathcal{E})$.

It will therefore suffice to show that this functor is a weak homotopy equivalence. To prove this, it suffices to show that it induces an equivalence on homotopy fibers taken over any point $(f : X_0 \to X_1) \in \mathrm{TwArr}(\mathcal{E})$. Unwinding the definitions, we wish to show that composition with f induces a weak homotopy equivalence

$$\mathcal{D}_{X_1} \to \mathcal{D}^+_{X_0};$$

this follows from Propositions 5 and 4.