The Whitehead Space (Lecture 30)

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Recall that our goal is to prove the following result:

Theorem 1. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_{\Delta}(X) \times_{\Omega^{\infty} A(X)} \{[X]\}$$

is a homotopy equivalence.

In the last two lectures, we introduced explicit combinatorial models for the spaces $K_{\Delta}(X)$ and $\Omega^{\infty} A(X)$. Our goal in this lecture is to do the same for $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$.

Definition 2. Let X be a simplicial set. We let $\mathcal{D}_X$ denote the subcategory of $(\mathrm{Set}_{\Delta})_{X/}$ whose objects are monomorphisms $i : X \hookrightarrow Y$ such that Y is obtained from X by adding finitely many simplices and i is a weak homotopy equivalence; the morphisms in $\mathcal{D}_X$ are diagrams

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & \xrightarrow{f} & Y' \end{array}$$

where f is cell-like. We let $W(X)$ denote the nerve of the category $\mathcal{D}_X$. We will soon see that $W(X)$ is equivalent to the 1st space of the Whitehead spectrum $\mathrm{Wh}(X)$. We will generally abuse notation by not distinguishing between $W(X)$ and its image in the ∞-category of spaces (obtained by choosing a fibrant replacement for $W(X)$).

Recall that $\mathcal{M}$ can be identified with the nerve of the category of finite simplicial sets and cell-like maps. Consequently, the next result should not be so surprising:

Theorem 3. *Let X be a finite simplicial set. Then there is a homotopy equivalence*

$$\mathcal{M} \times_{\mathcal{M}^h} \{|X|\} \simeq W(X),$$

which is natural with cell-like maps.

Corollary 4. *The construction $X \mapsto W(X)$ preserves weak homotopy equivalences.*

We will discuss Theorem 3 and Corollary 4 in the next lecture.

Note that $W(X)$ is a covariant functor of X, and that this functor commutes with filtered colimits. Consequently, $W(X)$ is formally determined by its restriction to finite simplicial sets.

The formation of pushouts over X determines a symmetric monoidal structure on the category $\mathcal{D}_X$, so that $W(X)$ inherits the structure of an E_{∞}-space. When X is finite, this E_{∞}-structure is compatible with the homotopy equialence of Theorem 3 (where we regard $\mathcal{M} \times_{\mathcal{M}^h} \{|X|\}$ as an E_{∞}-space as in Lecture 13). It follows that E_{∞}-structure on $W(X)$ is grouplike (when X is connected, we have $\pi_0 W(X) = \mathrm{Wh}(\pi_1 X)$). Since the functor $X \mapsto W(X)$ commutes with filtered colimits, this result extends formally to arbitrary simplicial sets:

Corollary 5. *For every simplicial set X, the E_∞-structure on $W(X)$ is grouplike.*

Let $\mathcal{C}_X$ denote the category studied in the previous two lectures: the objects of $\mathcal{C}_X$ are diagrams

$$\begin{array}{ccccc} & & Y & & \\ & \overset{i}{\nearrow} & & \searrow & \\ X & & \xrightarrow{\mathrm{id}} & & X \end{array}$$

where Y is obtained from X by adding finitely many simplices. We let $\mathcal{C}_X^h$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects where the map i is a weak homotopy equivalence.

Exercise 6. Show that we can regard $\mathcal{C}_X^h$ as a category with cofibrations and weak equivalences in two ways:

- We can take cofibrations in $\mathcal{C}_X^h$ to be monomorphisms and weak equivalences to be cell-like maps (we will denote this class of weak equivalences by s).
- We can take cofibrations in $\mathcal{C}_X^h$ to be monomorphisms and weak equivalences to be weak homotopy equivalences (we will denote this class of weak equivalences by h; note that it is the collection of *all* morphisms in $\mathcal{C}_X^h$).

We can apply Waldhausen's construction in either of these cases to obtain K-theory spaces $K(\mathcal{C}_X^h, h)$ and $K(\mathcal{C}_X^h, s)$. Note that each level of the simplicial category $hS_\bullet\, \mathcal{C}_X^h$ has an initial object, so that $K(\mathcal{C}_X^h, h)$ is contractible. The inclusion $\mathcal{C}_X^h \hookrightarrow \mathcal{C}_X$ induces maps on K-theory, giving us a commutative diagram

$$\begin{array}{ccc} K(\mathcal{C}_X^h, s) & \longrightarrow & K(\mathcal{C}_X, s) \\ \downarrow & & \downarrow \\ K(\mathcal{C}_X^h, h) & \longrightarrow & K(\mathcal{C}_X, h). \end{array}$$

We will deduce Theorem **??** from the following result, which we defer to a future lecture:

Theorem 7. *For every simplicial set X, the preceding diagram is a homotopy pullback square. In other words, we have a fiber sequence of spaces*

$$K(\mathcal{C}_X^h, s) \to K(\mathcal{C}_X, s) \to K(\mathcal{C}_X, h).$$

Corollary 8. *Let $X = \Delta^0$. Then the canonical map $K(\mathcal{C}_X, s) \to K(\mathcal{C}_X, h)$ is a homotopy equivalence.*

Proof. By virtue of Theorem 7 (and the evident surjectivity on π_0), it will suffice to show that $K(\mathcal{C}_X^h, s)$ is contractible. In fact, every level of the Waldhausen construction $sS_\bullet\, \mathcal{C}_X^h$ is weakly contractible: each of the categories $sS_n\, \mathcal{C}_X^h$ has a final object. $\square$

We have seen that there is a canonical homotopy equivalence $K(\mathcal{C}_X, h) \simeq \Omega^\infty A^{\mathrm{free}}(X)$, and that the functor $X \mapsto |K(\mathcal{C}_{X^{\Delta^\bullet}}, s)|$ is a homology theory. Corollary 8 implies that this is the homology theory associated to $A^{\mathrm{free}}(*) \simeq A(*)$:

Corollary 9. *For every simplicial set X, there is a canonical homotopy equivalence*

$$|K(\mathcal{C}_{X^{\Delta^\bullet}}, s)| \simeq \Omega^\infty(A(*) \wedge X_+) \simeq K_\Delta(X).$$

Note that since the assembly map

$$(A(*) \wedge X_+) \to A^{\mathrm{free}}(X)$$

is surjective on π_0, Theorem 7 implies that we have a fiber sequence of spectra

$$\Omega^{-\infty} K(\mathcal{C}_X^h, s) \to \Omega^{-\infty} K(\mathcal{C}_X, s) \to \Omega^{-\infty} K(\mathcal{C}_X, h).$$

Replacing X by $X^{\Delta^\bullet}$ and passing to the geometric realization, we obtain the following:

Corollary 10. *For every simplicial set X, there is a canonical fiber sequence of spectra*

$$|\Omega^{-\infty} K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \to A(*) \wedge X_+ \to A^{\mathrm{free}}(X).$$

Passing to 0th spaces (and noting that $\Omega^\infty A^{\mathrm{free}}(X)$ is a union of path components of $\Omega^\infty A(X)$), we obtain:

Corollary 11. *For every simplicial set X, there is a canonical fiber sequence of spaces*

$$|K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \to \Omega^\infty (A(*) \wedge X_+) \to \Omega^\infty A(X).$$

To understand the relationship between Corollary 11 and Theorem **??**, let us investigate the relationship between $K(\mathcal{C}_X^h, s)$ and the space $W(X)$. Let $s\,\mathcal{C}_X^h$ denote the subcategory of $\mathcal{C}_X^h$ whose morphisms are cell-like maps (that is, the first stage of the Waldhausen construction on $(\mathcal{C}_X^h, s)$). We have an evident forgetful functor $s\,\mathcal{C}_X^h \to \mathcal{D}_x$, which induces a diagram

$$K(\mathcal{C}_X^h, s) \leftarrow \mathrm{N}(s\,\mathcal{C}_X^h) \to W(X).$$

Replacing X by $X^{\Delta^\bullet}$ and passing to geometric realizations, we obtain a diagram

$$|K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)| \leftarrow |s\,\mathcal{C}^h_{X^{\Delta^\bullet}}| \to W(X).$$

Proposition 12. *If X is a Kan complex, then the above maps are homotopy equivalences. Consequently, we obtain a preferred homotopy equivalence*

$$W(X) \simeq |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|,$$

so that Corollary 11 yields a fiber sequence

$$W(X) \to \Omega^\infty (A(*) \wedge X_+) \to \Omega^\infty A(X).$$

Remark 13. It follows from Corollary 11 that the construction

$$X \mapsto |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$$

preserves weak homotopy equivalences in X. Consequently, the identification $W(X) \simeq |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$ of Proposition 12 persists even if we do not assume that X is a Kan complex.

Proposition 12 is really two separate statements (Proposition 14 and Proposition 15 below):

Proposition 14. *Let X be a Kan complex. Then the canonical map of simplicial spaces*

$$\theta : \mathrm{N}(s\,\mathcal{C}^h_{X^{\Delta^\bullet}}) \to S(X^{\Delta^\bullet})$$

induces a homotopy equivalence after geometric realization.

Proof. For every simplicial set X, let $\mathcal{E}_{X,\bullet}$ denote the simplicial category whose objects (in simplicial degree m) are diagrams

$$\begin{array}{ccc} & Y & \\ {}^{i}\nearrow & & \searrow \\ X & \xrightarrow{\delta} & X^{\Delta^m} \end{array}$$

where i is a trivial cofibration of simplicial sets which exhibits Y as obtained from X by adding finitely many simplices, and whose morphisms are cell-like maps. Then $\mathcal{E}_{X,0} \simeq s\,\mathcal{C}^h_X$, and we have a canonical map $\mathcal{E}_{X,\bullet} \to \mathcal{D}_X$ (where we identify $\mathcal{D}_X$ with a constant simplicial category). It follows that θ factors as a composition

$$\mathrm{N}(s\,\mathcal{C}^h_X) \xrightarrow{\theta'} |\,\mathrm{N}(\mathcal{E}_{X,\bullet})| \xrightarrow{\theta''} \mathrm{N}(\mathcal{D}_X).$$

We first claim that θ'' is a homotopy equivalence: this follows from the observation that for each object $Y \in \mathcal{D}_X$, the Kan complex of retractions from Y to X (which are the identity on X) is contractible, since X is a Kan complex and Y is weakly equivalent to X. It will therefore suffice to show that θ' induces a homotopy equivalence

$$|\,\mathrm{N}(s\,\mathcal{C}^h_{X^{\Delta^\bullet}})| \to ||\,\mathrm{N}(\mathcal{E}^h_{X^{\Delta^\bullet},\bullet})||.$$

Let $\mathcal{E}_\bullet$ denote the simplicial category given by $[n] \mapsto \mathcal{E}_{X^{\Delta^n},n}$: the objects of $\mathcal{E}_n$ are given by commutative diagrams

$$\begin{array}{ccc} & Y & \\ {}^{i}\nearrow & & \searrow^{f} \\ X^{\Delta^n} & \longrightarrow & X^{\Delta^n \times \Delta^n} \end{array}$$

where i is a trivial cofibration which exhibits Y as obtained from X^{Δ^n} by adding finitely many simplices and f classifies a map of simplicial sets $F : Y \times \Delta^n \times \Delta^n \to X$ whose restriction to $X^{\Delta^n} \subseteq Y$ is obtained by ignoring the second factor of Δ^n and using the evaluation map on the first. Let us abuse notation by identifying the objects of $\mathcal{E}_n$ with the pairs (Y, F).

Let $\mathcal{E}'_\bullet$ be the full simplicial subcategory of $\mathcal{E}_\bullet$ given in simplicial degree n by those objects (Y, F) where the map $F : Y \times \Delta^n \times \Delta^n \to X$ does not depend on the second factor of Δ^n. Unwinding the definitions, we we wish to show that the inclusion $\mathcal{E}'_\bullet \hookrightarrow \mathcal{E}_\bullet$ induces a weak homotopy equivalence $|\,\mathrm{N}\,\mathcal{E}'_\bullet\,| \hookrightarrow |\,\mathrm{N}\,\mathcal{E}_\bullet\,|$.

For each $n \geq 0$, let $r^+_n : \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n$ denote the map given on vertices by

$$r^+_n(i,j) = \begin{cases} (i,j) & \text{if } i \leq j \\ (i,i) & \text{otherwise,} \end{cases}$$

and let r^-_n denote the map given by

$$r^-_n(i,j) = \begin{cases} (i,j) & \text{if } j \leq i \\ (i,i) & \text{otherwise,} \end{cases}$$

Then r^-_n and r^+_n induce idempotent maps R^-_n and R^+_n from $Y \times \Delta^n \times \Delta^n$ to itself. We let $\mathcal{E}^+_n$ and $\mathcal{E}^-_n$ denote the full subcategories of $\mathcal{E}_n$ spanned by those pairs (Y, F) where F factors through R^-_n and R^+_n, respectively. Then $\mathcal{E}^-_\bullet, \mathcal{E}^+_\bullet$ are simplicial subcategories of $\mathcal{E}_\bullet$ whose intersection is $\mathcal{E}'_\bullet$. It will therefore suffice to show that the inclusions

$$|\,\mathrm{N}\,\mathcal{E}'_\bullet\,| \overset{\phi}{\hookrightarrow} |\,\mathrm{N}\,\mathcal{E}^+_\bullet\,| \overset{\psi}{\hookrightarrow} |\,\mathrm{N}\,\mathcal{E}_\bullet\,|$$

are weak homotopy equivalences. We will show that ψ is a weak homotopy equivalence by demonstrating that $R^+_\bullet$ is a deformation retraction: that is, there is a simplicial homotopy the identity map to $R^+_\bullet$ which is fixed on $\mathcal{E}^+_\bullet$. A dual construction will show that $R^-_\bullet$ is a deformation retraction from $\mathcal{E}_\bullet$ to $\mathcal{E}^-_\bullet$; we will leave as an exercise to the reader to verify that this deformation retraction carries $\mathcal{E}_\bullet^+$ to itself and therefore exhibits $\mathcal{E}'_\bullet = \mathcal{E}_\bullet^+ \cap \mathcal{E}_\bullet^-$ as a deformation retract of $\mathcal{E}_\bullet^+$ (which will prove that ϕ is a weak homotopy equivalence).

To construct the desired simplicial homotopy, we must exhibit for each map of finite linearly ordered sets $\alpha : [n] \to [1]$ a functor

$$\rho_\alpha : \mathcal{E}_n \to \mathcal{E}_n$$

which is the identity when α has the constant value 0 and agrees with R_n^+ when α has the constant value 1. We define ρ_α by the formula

$$\rho_\alpha(Y, F) = (Y, F \circ (\mathrm{id}_Y \times r_\alpha)),$$

where $r_\alpha : \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n$ is the map given on vertices by the formula

$$r_\alpha(i,j) = \begin{cases} (i,j) & \text{if } i \leq j \text{ or } \alpha(j) = 0. \\ (i,i) & \text{otherwise.} \end{cases}$$

□

The other half of Proposition 12 is contained in the following:

Proposition 15. *Let X be a Kan complex. Then the canonical map*

$$\rho : |s\, \mathcal{C}^h_{X^{\Delta^\bullet}}| \to |K(\mathcal{C}^h_{X^{\Delta^\bullet}}, s)|$$

is a homotopy equivalence.

Proof. Let $\mathrm{Bar}_\bullet(s\, \mathcal{C}^h_X)$ denote the simplicial space obtained by applying the bar construction (with respect to the E_∞-structure given by pushouts over X). It follows from Proposition 14 that $|s\, \mathcal{C}^h_{X^{\Delta^\bullet}}|$ is equivalent to $W(X)$ and is therefore equivalent to its own group completion (with respect to the natural E_∞-structure), so we can identify $|s\, \mathcal{C}^h_{X^{\Delta^\bullet}}|$ with the loop space of $|\mathrm{Bar}_\bullet(s\, \mathcal{C}^h_{X^{\Delta^\bullet}})|$. Under this identification, ρ arises from a map of bisimplicial categories

$$\mathrm{Bar}_\bullet(s\, \mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_\bullet\, \mathcal{C}^h_{X^{\Delta^\bullet}}.$$

To show that this map is a homotopy equivalence, it will suffice to show that for each $n \geq 0$, the map of simplicial categories

$$\rho_n : \mathrm{Bar}_n(s\, \mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}})$$

induces a homotopy equivalence (after passing to nerves and taking geometric realizations). Let us identify the objects of $sS_n(\mathcal{C}^h_{X^{\Delta^m}})$ with finite sequences of cofibrations

$$Y_1 \hookrightarrow \cdots \hookrightarrow Y_n$$

of simplicial sets over and under X^{Δ^m}, which are weak homotopy equivalent to X^{Δ^m} and obtained from X^{Δ^m} by adding finitely many simplices. The map ρ_n has a left homotopy inverse, given by a functor

$$\psi_n : sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \to (s\, \mathcal{C}^h_{X^{\Delta^\bullet}})^n$$

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n) \mapsto (Y_1, Y_2/Y_1, \ldots, Y_n/Y_{n-1}).$$

It will therefore suffice to show that ψ_n induces a homotopy equivalence (after taking nerves and passing to the geometric realizaiton). The proof proceeds by induction on n, the case $n \leq 1$ being trivial. To carry out the inductive step, it will suffice (by virtue of the inductive hypothesis and Propoistion 14) to show that the functor

$$\phi : sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \to sS_{n-1}(\mathcal{C}^h_{X^{\Delta^\bullet}}) \times \mathcal{D}_{X^{\Delta^\bullet}}$$

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n) \mapsto ((Y_1 \hookrightarrow \cdots Y_{n-1}), Y_n/Y_{n-1})$$

induces a homotopy equivalence (after taking nerves and passing to the geometric realization).

Let $\mathcal{E}_\bullet$ denote the simplicial category whose objects in degree m are diagrams

$$X^{\Delta^m} \to Y_1 \to \cdots \to Y_{n-1} \to Y_n \times X^{\Delta^m}$$

where each Y_i is obtained from Y_{i-1} by adding finitely many simplices and is weakly homotopy equivalent to Y_{i-1} (with the convention that $Y_0 = X^{\Delta^m}$), and the composite map induces the identity from X^{Δ^m} to itself, and whose morphisms are degreewise cell-like maps. More informally: $\mathcal{E}_\bullet$ is defined just like the simplicial category $sS_n(\mathcal{C}^h_{X^\bullet})$, except that we do not require that the simplicial set Y_n is equipped with a retraction back onto $X^{\Delta^\bullet}$. The map ϕ factors as a composition

$$sS_n(\mathcal{C}^h_{X^{\Delta^\bullet}}) \xrightarrow{\phi'} \mathcal{E}_\bullet \xrightarrow{\phi''} sS_{n-1}(\mathcal{C}^h_{X^{\Delta^\bullet}}) \times \mathcal{D}_{X^{\Delta^\bullet}}$$

Note that when $n = 1$, the map ϕ' coincides with the map studied in Proposition 14, and is therefore a homotopy equivalence (after taking nerves and passing to the geometric realization). Moreover, the proof of Proposition 14 can be adapted (with minor changes of notation) to prove that this assertion holds for all n. We are therefore reduced to proving that the map ϕ'' induces a homotopy equivalence. In fact, we claim that this holds even before passage to geometric realization: that is, for each m, the functor

$$\phi''_m : \mathcal{E}_m \to sS_{n-1}(\mathcal{C}^h_{X^{\Delta^m}}) \times \mathcal{D}_{X^{\Delta^m}}$$

induces a weak homotopy equivalence of nerves. Replacing X by X^{Δ^m}, we may assume that $m = 0$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}^0 & \longrightarrow & sS_{n-1}(\mathcal{C}^h_X) \times \mathcal{D}_X \\ & \searrow \quad \swarrow & \\ & sS_{n-1}(\mathcal{C}^h_X) & \end{array}$$

where the vertical maps are coCartesian fibrations. Consequently, to prove that the horizontal map is a weak homotopy equivalence, it will suffice to show that it induces a weak homotopy equivalence after taking the fiber over any object

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_{n-1}) \in sS_{n-1}(\mathcal{C}^h_X).$$

Unwinding the definitions, this amounts to the assertion that the retraction $Y_{n-1} \to X$ induces a weak homotopy equivalence

$$\mathcal{D}_{Y_{n-1}} \to \mathcal{D}_X,$$

which follows from Corollary 4. □

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*