Whitehead Torsion (Lecture 3)

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Let X be a space with the homotopy type of a CW complex. In the previous lecture, we studied the question of whether or not that CW could be chosen to be *finite*. More precisely, we saw that if a connected space X is finitely dominated (meaning that it behaves cohomologically like a finite CW complex), then there is an obstruction $\eta \in \widetilde{K}_0(\pi_1 X)$ which vanishes if and only if X has the homotopy type of a finite CW complex Y.

In this lecture, we will study the question of uniqueness. Suppose we are given two finite CW complexes Y and Y' equipped with homotopy equivalences

$$f : Y \to X \qquad g : X \to Y'.$$

Then $g \circ f$ is a homotopy equivalence from Y to Y'. One can ask if this homotopy equivalence can be "witnessed" entirely in the world of finite CW complexes.

In what follows, we use the term *CW complex* to refer to a space Y with a *specified* decomposition into open cells. For each integer $n \geq -1$, we let Y^n denote the n-skeleton of Y. Recall that a map of CW complexes $f : X \to Y$ is *cellular* if it carries each X^n into Y^n.

Construction 1. Let D^n denote the closed unit ball of dimension n and let $S^{n-1} = \partial D^n$ denote its boundary. We will regard S^{n-1} as decomposed into hemispheres S^{n-1}_- and S^{n-1}_+ which meet along the "equator" $S^{n-2} = S^{n-1}_- \cap S^{n-1}_+$.

Let Y be a CW complex equipped with a map $f : (S^{n-1}_-, S^{n-2}) \to (Y^{n-1}, Y^{n-2})$. Then the pushout $Y \amalg_{S^{n-1}_-} D^n$ has the structure of a CW complex which is obtained from Y by adding two more cells: an $(n-1)$-cell given by the image of the interior of S^{n-1}_+ (attached via the map $f|_{S^{n-2}} : S^{n-2} \to Y^{n-2}$) and an n-cell given by the image of the interior of D^n attached via the map

$$S^{n-1} = S^{n-1}_- \amalg_{S^{n-2}} S^{n-1}_+ \to Y^{n-1} \amalg_{S^{n-2}} S^{n-1}_+.$$

In this case, we will refer to the CW complex $Y \amalg_{S^{n-1}_-} D^n$ as an *elementary expansion* of Y, and to the inclusion map $Y \hookrightarrow Y \amalg_{S^{n-1}_-} D^n$ as an *elementary expansion*.

The hemisphere $S^{n-1}_- \subseteq D^n$ is a retract (even a deformation retract) of D^n. Composition with any retraction induces a (cellular) $c : Y \amalg_{S^{n-1}_-} D^n \to Y$, which we will refer to as an *elementary collapse*. Note that the homotopy class of c does not depend on the choice of retraction $D^n \to S^{n-1}_-$

Definition 2. Let $f : X \to Y$ be a map of CW complexes. We will say that f is a *simple homotopy equivalence* if it is homotopic to a finite composition

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n = Y,$$

where each f_i is either an elementary expansion or an elementary collapse.

We say that two finite CW complexes are *simple homotopy equivalent* if there exists a simple homotopy equivalence between them.

Example 3. Let X and Y be finite CW complexes and let $f : X \to Y$ be a continuous map. We let $M(f) = (X \times [0,1]) \amalg_{X \times \{1\}} Y$ denote the mapping cylinder of f. If f is a cellular map, then we can regard $M(f)$ as a finite CW complex (taking the cells of $M(f)$ to be the cells of Y together with cells of the form $e \times \{0\}$ and $e \times (0,1)$, where e is a cell of X). The inclusion $Y \hookrightarrow M(f)$ is always a simple homotopy equivalence: in fact, it can be obtained by a finite sequence of elementary expansions which simultaneously add pairs of cells $e \times \{0\}$ and $e \times (0,1)$ (where we add cells in order of increasing dimension).

Note that the map f is homotopic to a composition

$$X \simeq X \times \{0\} \xrightarrow{\iota} M(f) \xrightarrow{r} Y,$$

where r is the canonical retraction from $M(f)$ onto Y (which is homotopy inverse to the inclusion $Y \hookrightarrow M(f)$, and can be obtained by composing a finite sequence elementary collapses). It follows that f is a simple homotopy equivalence if and only if ι is a simple homotopy equivalence. Consequently, when we are studying the question of whether or not some map f is a simple homotopy equivalence, there is no real loss of generality in assuming that f is the inclusion of a subcomplex.

It is easy to see that any simple homotopy equivalence is a homotopy equivalence. One can ask whether the converse holds:

Question 4. Let $f : X \to Y$ be a homotopy equivalence between finite CW complexes. Is f a simple homotopy equivalence? If not, how can we tell?

To address Question 4, we will introduce an algebraic invariant (called the *Whitehead torsion*) which vanishes for simple homotopy equivalences, but not for all homotopy equivalences. First, we need a brief digression.

Definition 5. Let R be a ring (not necessarily commutative). For each integer $n \geq 0$, we let $\mathrm{GL}_n(R)$ denote the group of automorphisms of R^n as a right R-module (equivalently, the group of invertible n-by-n matrices with coefficients in R). Every automorphism α of R^n extends to an automorphism $\alpha \oplus \mathrm{id}_R$ of R^{n+1}; this construction yields inclusions

$$\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_3(R) \hookrightarrow \cdots$$

We let $\mathrm{GL}_\infty(R)$ denote the direct limit of this sequence, and we define $K_1(R)$ to be the abelianization of $\mathrm{GL}_\infty(R)$.

Remark 6. Let R be a commutative ring. For every integer n, the determinant gives a group homomorphism

$$\det : \mathrm{GL}_n(R) \to R^\times.$$

These maps are compatible as n varies and therefore determine a group homomorphism $\det : K_1(R) \to R^\times$. This map is split surjective (split by the canonical map $\mathrm{GL}_1(R) \to \mathrm{GL}_\infty(R) \to K_1(R)$). This map can be shown to be an isomorphism when R is a field or $R = \mathbf{Z}$, but it is not an isomorphism in general.

By construction, for any ring R we have a canonical homomorphism $\mathrm{GL}_n(R) \to K_1(R)$, which one can think of as a kind of "universal determinant".

Exercise 7. For every unit $x \in R^\times$, let $[x]$ denote the image of x under the composite map $\mathrm{GL}_1(R) \to \mathrm{GL}_\infty(R) \to K_1(R)$. Suppose that $\sigma \in \mathrm{GL}_n(R)$ is a permutation matrix. Show that the image of σ in $K_1(R)$ is given by $[\epsilon]$, where $\epsilon = \pm 1$ is the sign of the permutation σ.

Exercise 8. Let $g \in \mathrm{GL}_n(R)$. Let us say that g is *potentially upper triangular* if there exists a decomposition of R^n as a direct sum $P_1 \oplus P_2 \oplus \cdots \oplus P_m$ such that for each $x \in P_i$, we have

$$g(x) \in x + P_1 + \cdots + P_{i-1}.$$

Show that if g is potentially upper triangular, then the image of g in $K_1(R)$ vanishes.

Let R be a ring. A *based chain complex over* R is a bounded chain complex of R-modules

$$\cdots \to F_n \xrightarrow{d} F_{n-1} \xrightarrow{d} F_{n-2} \to \cdots$$

together with a choice of *unordered* basis for each F_m (so that each F_m is a free R-module). In this case, we let $\chi(F_*)$ denote the sum $\sum(-1)^m r_m$, where r_m denotes the cardinality of the basis of F_m. We will refer to $\chi(F_*)$ as the *Euler characteristic* of (F_*, d).

Warning 9. If R is a nonzero commutative ring, then the Euler characteristic $\chi(F_*)$ is independent of the choice of basis of the modules F_*. For a general noncommutative ring R, this need not be the case.

Exercise 10. Let (F_*, d) be a finite based chain complex which is *acyclic*: that is, the homology of (F_*, d) vanishes. Show that if R admits a nonzero homomorphism to a commutative ring, then $\chi(F_*, d) = 0$.

Let (F_*, d) be a based chain complex over R which is *acyclic*. Since each F_m is a free R-module, it then follows that the identity map $\mathrm{id} : F_* \to F_*$ is chain homotopic to zero: that is, there exists a map $h : F_* \to F_{*+1}$ satisfying $dh + hd = \mathrm{id}$. We let $F_{\mathrm{even}} = \bigoplus_n F_{2n}$ and $F_{\mathrm{odd}} = \bigoplus_n F_{2n+1}$.

Lemma 11. *In the situation above, the map $d + h : F_{\mathrm{even}} \to F_{\mathrm{odd}}$ is an isomorphism.*

Proof. We have $(d + h)(d + h) = d^2 + dh + hd + h^2 = \mathrm{id} + h^2$, which has an inverse given by the sum $1 - h^2 + h^4 - h^6 + \cdots$ (note that this sum is actually finite, since the chain complex F_* is bounded and h increases degrees). □

The specification of a basis for each F_m determines isomorphisms

$$F_{\mathrm{even}} \simeq R^a \qquad F_{\mathrm{odd}} \simeq R^b$$

for some integers $a, b \geq 0$, which are well-defined up to the action of permutation matrices.

Definition 12. Let $\widetilde{K}_1(R)$ denote the quotient of $K_1(R)$ by the subgroup $[\pm 1]$. If (F_*, d) is an acyclic based complex with $\chi(F_*) = 0$, we define the *torsion* of (F_*, d) to be the image of $d + h \in \mathrm{GL}_a(R)$ under the map $\mathrm{GL}_a(R) \to \mathrm{GL}_\infty(R) \to \widetilde{K}_1(R)$; by virtue of Exercise 7, this does not depend on the ordering of the basis elements of F_*. We will denote the torsion of (F_*, d) by $\tau(F_*)$.

Lemma 13. *In the situation of Definition 12, the torsion $\tau(F_*)$ is well-defined: that is, it does not depend on the choice of nullhomotopy h.*

Proof. Any other nullhomotopy of (F_*, d) has the form $h+e$, where $e : F_* \to F_{*+1}$ is a map satisfying $de+ed = 0$. We have already seen that $(d+h)^2 = 1+h^2$, so that we have $(d+h)^{-1} = (d+h)(1-h^2+h^4-h^6+\cdots)$. Multiplying by $(d + h + e)$, we obtain

$$\begin{aligned} (d+h+e)(d+h)^{-1} &= 1 + e(d+h)^{-1} \\ &= 1 + e(d+h)(1-h^2+h^4+\cdots) \\ &= 1 + ed + \text{ degree ¿ 0 }. \end{aligned}$$

Since (F_*, d) is a bounded acyclic chain complex of free modules, it is split exact: in particular, each F_n contains the group $Z_n = \ker(d : F_n \to F_{n-1})$ as a direct summand. Note that ed annihilates the group Z_n, and that $ed = -de$ carries F_n into Z_n. It follows that any map of the form $1 + ed +$ degree ¿ 0 is potentially upper triangular when regarded as an automorphism of F_{even}, and therefore has vanishing image in $K_1(R)$ (Exercise 8). □

Exercise 14. Suppose we are given a short exact sequence of finite based chain complexes

$$0 \to (F'_*, d') \to (F_*, d) \to (F''_*, d'') \to 0.$$

Assume that the chosen basis for each F_m consists of the images of the basis elements of F'_m together with preimages of the basis elements of each F''_m. Show that:

(*a*) If (F'_*, d') and (F''_*, d'') are acyclic, then (F_*, d) is acyclic.

(*b*) If $\chi(F'_*, d') = \chi(F''_*, d'') = 0$, then $\chi(F_*, d) = 0$.

(*c*) If conditions (*a*) and (*b*) hold, then $\tau(F_*, d) = \tau(F'_*, d')\tau(F''_*, d'')$ in $\widetilde{K}_1(R)$.

Definition 15. Let $f : X_* \to Y_*$ be a map of chain complexes over a ring R. The *mapping cone of* f is defined to be the chain complex

$$C(f)_* = X_{*-1} \oplus Y_*$$

with differential $d(x, y) = (-dx, f(x) + dy)$ Note that if X_* and Y_* are based complexes, then we can regard $C(f)_*$ as a based complex (where we fix some convention for how our bases should be ordered; we will not worry about this point).

Suppose that we have $\chi(X_*, d) = \chi(Y_*, d)$ and that f is a quasi-isomorphism (that is, it induces an isomorphism on homology). Then $\chi(C(f)_*, d) = 0$ and $C(f)_*$ is acyclic. We define the *torsion of* f to be the element $\tau(f) = \tau(C(f)_*, d) \in K_1(R)$.

Example 16. Let (F_*, d) be an acyclic based complex with $\chi(F_*) = 0$, and let f be the identity map from F_* to itself. Then the mapping cone $C(f)_*$ has an explicit nullhomotopy given by $(x, y) \mapsto (y, 0)$. Let us identify $C(f)_{\text{even}}$ and $C(f)_{\text{odd}}$ with F_*, so that $d + h$ is given by

$$(x, y) \mapsto (y - dx, x + dy).$$

This map is given by a permutation matrix modulo the filtration by degree, so we have $\tau(f) = 1 \in \widetilde{K}_1(R)$.

Let us now explain how to apply the preceding ideas. Suppose that X and Y are finite CW complexes and that we are given a homotopy equivalence $f : X \to Y$. For simplicity, we will assume that X and Y are connected (otherwise, we can analyze each connected component separately). We fix a base point $x \in X$ and set $G = \pi_1(X, x) \simeq \pi_1(Y, f(x))$. Let $\widetilde{Y}$ be a universal cover of Y and let $\widetilde{X} = X \times_Y \widetilde{Y}$ be the corresponding universal cover of X, so that G acts on $\widetilde{X}$ and $\widetilde{Y}$ by deck transformations. Let us further assume that f is a cellular map. Then f induces a map of cellular chain complexes

$$\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z}).$$

Note that we can regard $C_*(\widetilde{X}; \mathbf{Z})$ and $C_*(\widetilde{Y}; \mathbf{Z})$ as chain complexes of free $\mathbf{Z}[G]$-modules, with basis elements in bijection with the cells of X and Y respectively. Since f is a homotopy equivalence, the map λ is a quasi-isomorphism. We may therefore consider the torsion $\tau(\lambda) \in \widetilde{K}_1(\mathbf{Z}[G])$. However, it is not quite well-defined: in order to extract an element of $C_*(\widetilde{X}; \mathbf{Z})$ from a cell $e \subseteq X$, we need to choose a cell of $\widetilde{X}$ lying over e (which is ambiguous up to the action of G) and an orientation of the cell e (which is ambiguous up to a sign). This motivates the following:

Definition 17. Let G be a group. The *Whitehead group* $\mathrm{Wh}(G)$ of G is the quotient of $K_1(\mathbf{Z}[G])$ by elements of the form $[\pm g]$, where $g \in G$.

If $f : X \to Y$ is a cellular homotopy equivalence of connected finite CW complexes, we define the *Whitehead torsion* $\tau(f) \in \mathrm{Wh}(G)$ to be the image in $\mathrm{Wh}(G)$ of the torsion of the induced map

$$\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z}).$$

We will continue our discussion of the Whitehead torsion in the next lecture.