Another Model of the Assembly Map II (Lecture 29)

November 12, 2014

Let X be a simplicial set. In the previous lecture, we introduced the category $\mathcal{C}_X$ whose objects are simplicial sets Y over and under X, which are obtained from X by adding finitely many simplices. We can regard $\mathcal{C}_X$ as a category with cofibrations and weak equivalences (where the latter is given by the collection s of cell-like maps), and we proved that the construction

$$F(X) = \Omega^{-\infty} K(\mathcal{C}_X, s)$$

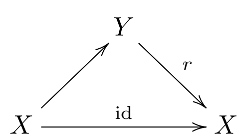
has the property that $\widehat{F}(X) = |F(X^{\Delta^\bullet})$ is a homology theory (that is, it induces a colimit-preserving functor from spaces to spectra). There is a natural map $\widehat{F}(X) \to A(X)$; when X is finite and nonsingular, this comes from a composite map

$$F(X) \to \Omega^{-\infty} K(\mathrm{Shv}_{PL}(X)^{\mathrm{op}}) \to \Omega^{-\infty} K_{\Delta}(X) \to A(X)$$

where the first map is obtained from a functor

$$\lambda : \mathcal{C}_X \to \mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}$$

which assigns to each retraction diagram



the constructible sheaf given by the cofiber of the unit map $\underline{S}_X \to r_* \underline{S}_Y$. To verify that this construction yields a well-defined map on K-theory, we observe that if $Y, Y' \in \mathcal{C}_X$ are related by a cell-like map $Y \to Y'$, then the constant sheaf $\underline{S}_{Y'}$ can be identified with the direct image of the constant sheaf $\underline{S}_Y$, so that the induced map $\lambda(Y') \to \lambda(Y)$ is an equivalence in $\mathrm{Shv}_{PL}(|X|)$.

Consider the functor $\mu : \mathrm{Shv}_{PL}(|X|)^{\mathrm{op}} \to (\mathrm{Sp}^X)^c$ (here $(\mathrm{Sp}^X)^c$ denotes the ∞-category of compact objects of Sp^X) characterized by the formula

$$\mathrm{Map}_{\mathrm{Sp}^X}(\mu(\mathcal{F}), \mathcal{G}) = \Gamma(|X|, \mathcal{F} \wedge \mathcal{G}).$$

Unwinding the definitions, we see that $\mu \circ \lambda : \mathcal{C}_X \to (\mathrm{Sp}^X)^c$ is given by the formula $Y \mapsto r_! \underline{S}_Y$, where $r_! : \mathrm{Sp}^Y \to \mathrm{Sp}^X$ is the homological pushforward on local systems. Here we have a bit more flexibility: in order to ensure that a map $Y \to Y'$ in $\mathcal{C}_X$ induces an equivalence $(\mu \circ \lambda)(Y) \to (\mu \circ \lambda)(Y')$ in Sp^X, it is sufficient to assume that $Y \to Y'$ is a weak homotopy equivalence.

Exercise 1. Let h be the collection of all weak homotopy equivalences in $\mathcal{C}_X$. Show that $(\mathcal{C}_X, h)$ satisfies the axioms for a category with cofibrations and weak equivalences (where the cofibrations, as before, are given by the monomorphisms).

The above analysis supplies a diagram of infinite loop spaces

$$\begin{array}{ccc} K(\mathcal{C}_X, s) & \longrightarrow & K(\mathcal{C}_X, h) \\ \downarrow & & \downarrow \theta \\ K_\Delta(X) & \longrightarrow & \Omega^\infty A(X) \end{array}$$

which commutes up to canonical homotopy and depends functorially on X. Our goal in this lecture is to show that the right vertical map is close to being a homotopy equivalence. To this end, recall that $(\mathrm{Sp}^X)^c$ can be identified with the Spanier-Whitehead ∞-category of the ∞-category $(\mathcal{S}_*^X)^c \simeq \mathcal{S}^c_{X//X}$, where $\mathcal{S}_{X//X}$ denotes the ∞-category of spaces over and under X and $\mathcal{S}^c_{X//X}$ is the full subcategory of $\mathcal{S}_{X//X}$ spanned by the compact objects. Let $\mathcal{S}^{\mathrm{fin}}_{X//X} \subseteq \mathcal{S}^c_{X//X}$ denote the full subcategory spanned by those objects Y which can be obtained from X by attaching finitely many cells. Then θ is the map on K-theory induces by the composition

$$\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X} \subseteq \mathcal{S}^c_{X//X} \to (\mathrm{Sp}^X)^c.$$

We have seen that the map $K(\mathcal{S}^c_{X//X}) \to K((\mathrm{Sp}^X)^c) \simeq \Omega^\infty A(X)$ is a homotopy equivalence, and that the map $K(\mathcal{S}^{\mathrm{fin}}_{X//X}) \to K(\mathcal{S}^c_{X//X})$ exhibits the domain as a union of connected components of the target.

Notation 2. Let X be a space. We let $A^{\mathrm{free}}(X)$ denote the spectrum given by $\Omega^{-\infty} K(\mathcal{S}^{\mathrm{fin}}_{X//X})$. Then $A^{\mathrm{free}}(X)$ is a connective spectrum whose homotopy groups are given by

$$\pi_i A^{\mathrm{free}}(X) = \begin{cases} \pi_i A(X) & \text{if } i > 0 \\ \mathrm{H}_0(X; \mathbf{Z}) & \text{if } i = 0. \end{cases}$$

(note: this is the definition of $A(X)$ that appears in Waldhausen's paper).

Our next goal is to prove:

Proposition 3. *Let X be a simplicial set. Then the natural map $\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X}$ induces homotopy equivalences*

$$K(\mathcal{C}_X, h) \to K(\mathcal{S}^{\mathrm{fin}}_{X//X})$$

$$\Omega^{-\infty} K(\mathcal{C}_X, h) \to A^{\mathrm{free}}(X).$$

Note that the domain and codomain of the map appearing in the statement of Proposition 3 can be identified with the geometric realization of simplicial spaces obtained from Waldhausen's construction. It will therefore suffice to show that that the map $\mathcal{C}_X \to \mathcal{S}^{\mathrm{fin}}_{X//X}$ induces an equivalence in each simplicial degree. In other words, Proposition 3 is a consequence of the following more precise assertion:

Proposition 4. *Let X be a simplicial set and let $n \geq 0$ be an integer. Then the natural map*

$$hS_n \, \mathcal{C}_X \to S_n \, \mathcal{S}^{\mathrm{fin}}_{X//X}$$

is a weak homotopy equivalence (here we regard the left hand side as the nerve of a category and the right hand side as a Kan complex).

The proof of Proposition 4 will proceed by induction on n. The case $n = 0$ is trivial (since both sides are contractible), so let us assume $n > 0$. Let us identify the objects of $hS_n \, \mathcal{C}_X$ with chains of cofibrations

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_n$$

of simplicial sets over and under X. There is a canonical map $\tau : hS_n\,\mathcal{C}_X \to hS_{n-1}\,\mathcal{C}_X$ obtained by "forgetting" Y_n (one of the face maps appearing in the simplicial category $hS_\bullet\,\mathcal{C}_X$). Similarly, we have a forgetful map $S_n\,\mathcal{S}^{\text{fin}}_{X//X} \to S_{n-1}\,\mathcal{S}^{\text{fin}}_{X//X}$ which is a Kan fibration. These maps fit into a commutative diagram

$$\begin{array}{ccc} hS_n\,\mathcal{C}_X & \longrightarrow & S_n\,\mathcal{S}^{\text{fin}}_{X//X} \\ \downarrow{\scriptstyle \tau} & & \downarrow \\ hS_{n-1}\,\mathcal{C}_X & \longrightarrow & S_{n-1}\,\mathcal{S}^{\text{fin}}_{X//X}. \end{array}$$

The inductive hypothesis implies that the lower horizontal map is a weak homotopy equivalence. Consequently, to prove Proposition 4, it will suffice to show that the diagram induces a weak homotopy equivalence after taking homotopy fibers in the vertical direction. Fix an object $\vec{Y} = (Y_1 \to \cdots \to Y_{n-1})$ in $hS_{n-1}\,\mathcal{C}_X$. We will show that the category

$$(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\}$$

is weakly homotopy equivalent to the Kan complex

$$S_n\,\mathcal{S}^{\text{fin}}_{X//X} \times_{S_{n-1}\,\mathcal{S}^{\text{fin}}_{X//X}} \{\vec{Y}\}.$$

It will then follow that every map $\vec{Y} \to \vec{Y}'$ in $hS_{n-1}\,\mathcal{C}_X$ induces a weak homotopy equivalence

$$(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\} \to (hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}'\}.$$

Applying Quillen's Theorem B (and the observation that τ is a coCartesian fibration), it follows that $(hS_n\,\mathcal{C}_X) \times_{(hS_{n-1}\,\mathcal{C}_X)} \{\vec{Y}\}$ can be identified with the homotopy fiber of τ over $\vec{Y}$, thereby completing the proof of the inductive step. We are therefore reduced to proving the following lemma (applied in the case $Z = Y_{n-1}$):

Lemma 5. *Let $f : Z \to X$ be a map of simplicial sets. Let $\mathcal{C}_f$ denote the category whose objects are diagrams of simplicial sets*

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle j}\nearrow & & \searrow & \\ Z & & \xrightarrow{\quad f \quad} & & X \end{array}$$

where j is a cofibration and Y is obtained from Z by adding only finitely many simplices, and whose morphisms are weak homotopy equivalences. Let $\mathcal{S}^{\text{fin}}_{Z//X}$ denote the ∞-category given by the full subcategory of $\mathcal{S}_{Z//X}$ spanned by those objects Y which can be obtained from Z by attaching finitely many cells. Then the canonical map

$$\nu : \mathcal{C}_f \to (\mathcal{S}^{\text{fin}}_{Z//X})^{\simeq}$$

is a weak homotopy equivalence of simplicial sets.

Proof. Let us compute the homotopy fiber of ν over a point $\eta \in (\mathcal{S}^{\text{fin}}_{Z//X})^{\simeq}$. Let us represent η by a diagram of simplicial sets

$$\begin{array}{ccccc} & & W & & \\ & {\scriptstyle j}\nearrow & & \searrow{\scriptstyle q} & \\ Z & & \xrightarrow{\quad f \quad} & & X \end{array}$$

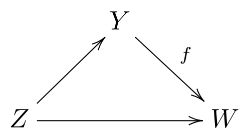
where j is a cofibration and q is a Kan fibration. Then the homotopy fiber $\nu^{-1}\{\eta\}$ can be identified with the homotopy colimit

$$\varinjlim_{Y \in \mathcal{C}_f} \underline{\text{Hom}}(Y, W),$$

where $\underline{\mathrm{Hom}}(Y, W)$ denotes the Kan complex parametrizing maps from Y to W in $(\mathrm{Set}_\Delta)_{Z//X}$ which are weak homotopy equivalences. It follows that $\nu^{-1}\{\eta\}$ can be identified with the geometric realization of a simplicial space which is given in degree m by the homotopy colimit

$$\varinjlim_{Y \in \mathcal{C}_f} \underline{\mathrm{Hom}}(Y, W)_m.$$

It will therefore suffice to show that this homotopy colimit is contractible for each m. Replacing W by $W^{\Delta^m} \times_{X^{\Delta^m}} X$, we can reduce to the case where $m = 0$. In this case, the homotopy colimit can be identified with the nerve of the category $\mathcal{D}$ whose objects are commutative diagrams



where Y is obtained from Z by adjoining finitely many simplices and the map f is a homotopy equivalence. It will therefore suffice to show that $\mathcal{D}$ is weakly contractible. In fact, we claim that $\mathcal{D}$ is filtered. Note that $\mathcal{D}$ is a full subcategory of a filtered category $\mathcal{D}^+$, where we drop the requirement that the map f be a weak homotopy equivalence. To prove that $\mathcal{D}$ is also filtered, it will suffice to verify that for every object $Y \in \mathcal{D}^+$ there exists a morphism $Y \to Y'$ where $Y' \in \mathcal{D}$. We are therefore reduced to proving the following general assertion about simplicial sets:

Lemma 6. *Let $g : Y \to W$ be a map of simplicial sets. Suppose that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching finitely many cells. Then g factors as a composition*

$$Y \to Y' \xrightarrow{f} W$$

where f is a weak homotopy equivalence and Y' is obtained from Y by adding finitely many simplices.

Proof. For simplicity, let us assume that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching a single n-cell (the proof in the general case is similar). This n-cell is attached via a map $S^{n-1} \to |Y|$, which can be obtained as the geometric realization of a map of simplicial sets $A \to Y$ where A is some subdivision $\partial \Delta^n$. The map $|Y| \amalg_{S^{n-1}} D^n \to |Z|$ determines a nullhomotopy h of the composite map

$$A \to Y \to Z$$

after geometric realization. Replacing A by a subdivision if necessary, we may assume that the nullhomotopy h arises from a nullhomotopy in the category of simplicial sets. For $n \gg 0$, we may assume that h arises from a simplicial nullhomotopy of the composite map

$$A \to Y \to Z \to \mathrm{Ex}^n Z.$$

We can then take

$$Y' = Y \amalg_{\mathrm{Sd}^n A} \mathrm{Sd}^n (A \times \Delta^1) \amalg_{\mathrm{Sd}^n A} \Delta^0.$$

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References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*