Another Model of the Assembly Map (Lecture 28)

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Our goal over the next few lectures is to prove the following result:

Theorem 1. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}$$

is a homotopy equivalence.

The proof will follow [1]. Our first goal is to construct another model for the domain of the assembly map. This will require us to consider a slightly more general version of Waldhausen's construction.

Definition 2. Let $\mathcal{C}$ be an ∞-category with cofibrations. A collection of morphisms w in $\mathcal{C}$ is a *class of weak equivalences* if it satisfies the following conditions:

(1) The collection of morphisms w contains all equivalences and is closed under composition.

(2) If we are given a diagram

$$\begin{array}{ccccc} X & \longleftarrow & Y & \xrightarrow{f} & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & Y' & \xrightarrow{f'} & Z' \end{array}$$

where the vertical maps belong to w and both f and f' are cofibrations, then the induced map $X \amalg_Y Z \to X' \amalg_{Y'} Z'$ also belongs to w.

For each $n \geq 0$, we let $wS_n \mathcal{C}$ denote the subcategory of $\mathrm{Gap}([n], \mathcal{C})$ containing all objects, whose morphisms are maps of $[n]$-gapped objects $X \to Y$ with the property that each of the maps $X(i,j) \to Y(i,j)$ belongs to w. Then $wS_\bullet \mathcal{C}$ is a simplicial ∞-category. We let $K(\mathcal{C}, w)$ denote the loop space $\Omega|wS_\bullet \mathcal{C}|$.

When w is the collection of all equivalences in $\mathcal{C}$, Definition 2 reduces to our previous notion of K-theory for an ∞-category with cofibrations. We will be primarily interested in the case where $\mathcal{C}$ is an ordinary category.

Example 3. Let X be a simplicial set. Let Set_Δ denote the ordinary category of simplicial sets, and let $(\mathrm{Set}_\Delta)_{X//X}$ denote the category of simplicial sets over and under X (so that the objects of $(\mathrm{Set}_\Delta)_{X//X}$ are triples (Y, i, r) where Y is a simplicial set, $i : X \to Y$ is a morphism of simplicial sets, and $r : Y \to X$ is a morphism satisfying $r \circ i = \mathrm{id}_X$). We let $\mathcal{C}_X$ denote the full subcategory of $(\mathrm{Set}_\Delta)_{X//X}$ spanned by those objects where Y is obtained from X by adding finitely many simplices.

We will regard $\mathcal{C}_X$ as an ∞-category with cofibrations, where the cofibrations are given by monomorphisms of simplicial sets. We will say that a morphism $f : Y \to Y'$ in $\mathcal{C}_X$ is *cell-like* if the induced map $|Y| \to |Y'|$ has contractible fibers (this is a slight variant on our earlier definition of cell-like, since the simplicial sets Y and Y' may not be finite; note however that the inverse image of any simplex in Y' is a finite simpicial set). It is not difficult to see that this collection of maps satisfies the requirements of Definition 2. We will denote this collection of morphisms by s.

For every simplicial set X, let $F(X)$ denote the connective spectrum $\Omega^{-\infty} K(\mathcal{C}_X, s)$.

Remark 4 (Functoriality)**.** If $f : X \to X'$ is a map of simplicial sets, then the construction $Y \mapsto Y \amalg_X X'$ induces a functor from $\mathcal{C}_X$ to $\mathcal{C}_{X'}$ which preserves cofibrations and cell-like maps. It follows that f induces a map of spectra $F(X) \to F(X')$. In other words, we can regard F as a functor from the ordinary category of simplicial sets to the ∞-category Sp of spectra.

Exercise 5. Let $f : X \to X'$ be a map of finite simplicial sets. Show that the construction $Y' \mapsto X \times_{X'} Y'$ determines a functor $\mathcal{C}_{X'} \to \mathcal{C}_X$ which preserves cofibrations and weak equivalences, and therefore induces a map $f^* : F(X') \to F(X)$. If f is cell-like, show that the map f^* is homotopy inverse to the map $F(X) \to F(X')$ of Remark 4. (Argue that for $Y \in \mathcal{C}_X$ and $Y' \in \mathcal{C}_{X'}$, the unit and counit maps

$$Y \to X \times_{X'} (Y \amalg_X X')$$

$$(X \times_{X'} Y') \amalg_X X' \to Y'$$

are cell-like).

Corollary 6. *The functor* $F : \mathrm{Set}_\Delta \to \mathrm{Sp}$ *is a left Kan extension of its restriction to finite nonsingular simplicial sets.*

Proof. It is easy to see that F commutes with filtered colimits and is therefore a left Kan extension of its restriction to finite simplicial sets. To complete the proof, it will suffice to show that for every finite simplicial set X, the canonical map $\varinjlim F(X') \to F(X)$ is a homotopy equivalence, where the colimit is taken over the category $\mathcal{E}$ of all nonsingular finite simplicial sets X' with a map $f : X' \to X$. Choose a cell-like map $\widetilde{X} \to X$, where X is nonsingular. Using Exercise 5, we can reduce to showing that the map

$$\theta : \varinjlim_{X' \in \mathcal{E}} F(X' \times_X \widetilde{X}) \to F(\widetilde{X})$$

is an equivalence. Let $\mathcal{E}'$ denote the category of finite nonsingular simplicial sets with a map to $\widetilde{X}$, so that the construction $X' \mapsto X' \times_X \widetilde{X}$ induces a functor $\mathcal{E} \to \mathcal{E}'$. This functor has a left adjoint and is therefore left cofinal. We may therefore identify θ with the natural map $\varinjlim_{\widetilde{X}' \in \mathcal{E}'} F(\widetilde{X}') \to F(\widetilde{X})$, which is clearly an equivalence because $\mathcal{E}'$ contains $\widetilde{X}$ as a final object. $\square$

Remark 7. For every finite nonsingular simplicial set Y, let $\underline{S}_Y$ denote the constant sheaf on the polyhedron $|Y|$ taking the value S.

Let X be a finite nonsingular simplicial set. For each object $Y \in \mathcal{C}_X$, let $\lambda(Y)$ denote the fiber of the canonical map

$$\underline{S}_X \to \varprojlim_{i:\Delta^n \to Y} (r \circ i)_* \underline{S}_{\Delta^n},$$

where $r : Y \to X$ denotes the structural retraction. Then λ determines a functor from the ordinary category $\mathcal{C}_X$ to the ∞-category $\mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}$, which preserves pushouts by cofibrations and carries cell-like maps to equivalences. Passing to K-theory, we obtain a map

$$F(X) \to \Omega^{-\infty} K(\mathrm{Shv}_{PL}(|X|)^{\mathrm{op}}).$$

Let us abuse notation by regarding $\Omega^{-\infty} K_\Delta$ as a functor from the *ordinary* category of simplicial sets to the ∞-category of spectra, so that the above construction gives a map

$$F(X) \to \Omega^{-\infty} K_\Delta(X).$$

Using Corollary 6, we see that this construction formally extends (in an essentially unique way) to the case where the simplicial set X is arbitrary.

The map $F(X) \to \Omega^{-\infty} K_\Delta(X)$ is generally not an equivalence: the left hand side is not homotopy invariant, but the right hand side is. However, we can attempt to correct that as follows:

Definition 8. Let G be a functor from the category of simplicial sets to the ∞-category of spectra. For every simplicial set X, the construction $[n] \mapsto X^{\Delta^n}$ determines a simplicial object of Set_Δ. We define $\widehat{G} : \mathrm{Set}_\Delta \to \mathrm{Sp}$ by the formula

$$\widehat{G}(X) = |G(X^{\Delta^\bullet})|.$$

Note that we have an evident natural transformation $G \to \widehat{G}$.

Exercise 9. Let G be as in Definition **??**. Show that if a map of simplicial sets $X \to X'$ is a simplicial homotopy equivalence, then the induced map $\widehat{G}(X) \to \widehat{G}(X')$ is an equivalence of spectra.

Warning 10. It is not generally true that the functor $\widehat{G}$ carries weak homotopy equivalences of simplicial sets to equivalences in Sp. However, it will be true for the functors we are interested in.

Remark 11. In the situation of Definition 8, suppose that G carries simplicial homotopy equivalences to homotopy equivalences of spectra. Then the simplicial object $G(X^{\Delta^\bullet})$ is essentially constant, so the natural transformation $G \to \widehat{G}$ is an equivalence.

The natural transformation $F(X) \to \Omega^{-\infty} K_\Delta(X)$ constructed above yields a natural map

$$\widehat{F}(X) \to \widehat{\Omega^{-\infty} K_\Delta}(X) \simeq \Omega^{-\infty} K_\Delta(X),$$

where the second equivalence follows from Remark 11. We will soon show that this map is an equivalence. The main step will be the verify the following:

Proposition 12. *The functor $\widehat{F}$ is a homology theory. In other words:*

(a) *The functor $\widehat{F}$ carries weak homotopy equivalences of simplicial sets to equivalences in $\mathcal{S}$.*

(b) *The functor $\widehat{F}$ commutes with filtered colimits.*

(c) *The spectrum $\widehat{F}(\emptyset)$ vanishes*

(d) *Given a pushout diagram of simplicial sets*

$$\begin{array}{ccc} X_{01} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X, \end{array}$$

where the horizontal maps are monomorphisms, the diagram of spectra

$$\begin{array}{ccc} \widehat{F}(X_{01}) & \longrightarrow & \widehat{F}(X_0) \\ \downarrow & & \downarrow \\ \widehat{F}(X_1) & \longrightarrow & \widehat{F}(X) \end{array}$$

is a pushout square.

Remark 13. Part (a) of Proposition 12 implies that $\widehat{F}$ factors through a functor $\mathcal{S} \to \mathrm{Sp}$, and parts ($b$), ($c$), and ($d$) assert that this functor preserves colimits.

Remark 14. Assuming Proposition 12, the assertion that the natural map $\widehat{F}(X) \to \Omega^{-\infty} K_\Delta(X)$ is an equivalence can be reduced to the case where X has a single point.

The proof of Proposition 12 can be broken into two parts, one of which is entirely formal and the other of which depends on the details of our situation:

Lemma 15. *The functor F satisfies conditions (b), (c), and (d) of Proposition 12.*

Lemma 16. *Let $G : \mathrm{Set}_\Delta \to \mathrm{Sp}$ be a functor which satisfies conditions (b), (c), and (d) of Proposition 12. Then $\widehat{G}$ satisfies conditions (a), (b), (c), and (d) of Proposition 12.*

Proof of Lemma 15. Assertions (b) and (c) are trivial. Let us focus on (d). Suppose we are given a pushout diagram of simplicial sets σ :

$$\begin{array}{ccc} X_{01} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X, \end{array}$$

where the horizontal maps are monomorphisms. Let $\mathcal{C}_{X_0,X_{01}}$ denote the full subcategory of $\mathcal{C}_{X_0}$ spanned by those objects Y for which the projection map $Y \times_{X_0} X_{01}$ is an isomorphism. Then any object $Y \in \mathcal{C}_{X_0}$ sits in a canonical cofiber sequence

$$X_{01} \times_{X_0} Y \to Y \to Y'$$

where $Y' \in \mathcal{C}_{X_0,X_{01}}$. Using a variant of the additivity theorem proved in Lecture 17, we obtain a homotopy equivalence

$$K(\mathcal{C}_{X_0}, s) \simeq K(\mathcal{C}_{X_{01}}, s) \times K(\mathcal{C}_{X_0,X_{01}}, s).$$

Similarly, we have a homotopy equivalence

$$K(\mathcal{C}_X, s) \simeq K(\mathcal{C}_{X_1}, s) \times K(\mathcal{C}_{X,X_1}, s).$$

We conclude by observing that because σ is a pushout square, the map $X_0 \to X$ induces an equivalence of categories $\mathcal{C}_{X_0,X_{01}} \to \mathcal{C}_{X,X_1}$. □

Proof of Lemma 16. Let $\mathbf{\Delta}$ denote the category of simplices, which we regard as a full subcategory of the category Set_Δ of simplicial sets and also the larger ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ of simplicial spaces. Let $G_0 = G|_{\mathbf{\Delta}}$ and let $\overline{G}_0 : \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Sp}$ be a left Kan extension of G_0. Then $\overline{G}_0$ preserves small colimits. Consequently, the functor $\overline{G}_0|_{\mathrm{Set}_\Delta}$ satisfies conditions (b), (c), and (d) of Proposition 12. The universal property of $\overline{G}_0$ supplies a natural transformation

$$\alpha : \overline{G}_0|_{\mathrm{Set}_\Delta} \to G.$$

By construction, α is an equivalence when evaluated on simplices. Since the domain and codomain of α both satisfy (c) and (d), it follows by a simple induction that α is an equivalence when evaluated on any finite simplicial set. Since the domain and codomain of α both satisfy (b), we can conclude that α is an equivalence when evaluated on *any* simplicial set: that is, the functor G is a left Kan extension of its restriction to $\mathbf{\Delta}$. We now compute

$$\begin{aligned} \widehat{G}(X) &\simeq \varinjlim_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} G(X^{\Delta^m}) \\ &\simeq \varinjlim_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} \varinjlim_{\Delta^n \to X^{\Delta^m}} G(\Delta^n) \\ &\simeq \varinjlim_{[n] \in \mathbf{\Delta}} \varinjlim_{\Delta^m \to X^{\Delta^n}} G(\Delta^n). \end{aligned}$$

Since the diagonal map $X \to X^{\Delta^n}$ is a simplicial homotopy equivalence, we can rewrite the latter colimit as

$$\begin{aligned} \varinjlim_{[n] \in \mathbf{\Delta}} \varinjlim_{\Delta^m \to X} G(\Delta^n) &\simeq \varinjlim_{\Delta^m \to X} \varinjlim_{[n] \in \mathbf{\Delta}} G(\Delta^n) \\ &\simeq \varinjlim_{\Delta^m \to X} G(\Delta^0) \\ &\simeq G(\Delta^0) \wedge X_+. \end{aligned}$$

□

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*