The Assembly Map (Lecture 24)

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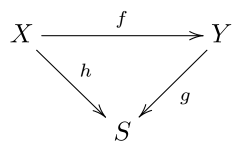
We begin this lecture by proving one more basic property of universal local acyclicity:

Proposition 1. *Suppose we are given a commutative diagram of finite polyhedra*



Suppose that g is a fibration and let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$. Then $\mathcal{F}$ is ULA over Y if and only if the following conditions are satisfied:

(1) *The sheaf $\mathcal{F}$ is ULA over S.*

(2) *For each point $s \in S$, the restriction $\mathcal{F}_s = \mathcal{F}|_{X_s}$ is ULA over Y_s.*

Example 2. When $X = Y$, Proposition 1 asserts that $\mathcal{F}$ is locally constant if and only if it is ULA over S and locally constant along each fiber X_s. In particular, if $\mathcal{F}$ is locally constant then it is ULA over S, as we saw in the previous lecture.

Proof of Proposition 1. Since the property of being ULA is stable under base change, condition (2) is clearly necessary. Let us therefore assume that $\mathcal{F}$ satisfies (2) and show that $\mathcal{F}$ is ULA over Y if and only if it is ULA over Z.

Fix compatible triangulations $\Sigma(X)$, $\Sigma(Y)$, and $\Sigma(S)$ of X, Y, and S such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Consider a simplex $\theta_0 \in \Sigma(X)$, and let $\sigma_0 \in \Sigma(Y)$ and $\tau_0 \in \Sigma(S)$ be its images in Y and S, respectively. Let $\tau \in \Sigma(S)$ be a simplex containing τ_0, let $P = \{\theta \in \Sigma(X) : \theta_0 \subseteq \theta, h(\theta) = \tau\}$ and let $Q = \{\sigma \in \Sigma(Y) : \sigma_0 \subseteq \sigma, g(\sigma) = \tau\}$. The map f determines a map of posets $P \to Q$. For each $\sigma \in Q$, let P_σ denote the inverse image of σ in P. We claim the following:

($*$) The right Kan extension of $\mathcal{F}|_P$ along the map $P \to Q$ is a locally constant functor (that is, it sends inequalities in Q to equivalences in $\mathcal{C}$).

Assuming ($*$), we can complete the proof as follows. Let $\mathcal{G}$ be the right Kan extension of $\mathcal{F}|_P$ along the map $P \to Q$. Since g is a fibration, the partially ordered set Q is weakly contractible. It follows from ($*$) that for any element $\sigma \in Q$, the canonical map $\varprojlim \mathcal{G} \to \mathcal{G}(\sigma)$ is an equivalence in $\mathcal{C}$. Rewriting this in terms of $\mathcal{F}$, we conclude that the canonical map

$$\varprojlim_{\theta \in P} \mathcal{F}(\theta) \to \varprojlim_{\theta \in P_\sigma} \mathcal{F}(\theta)$$

is an equivalence. It follows that the natural maps

$$\alpha : \mathcal{F}(\theta_0) \to \varprojlim_{\theta \in P} \mathcal{F}(\theta)$$

$$\beta : \mathcal{F}(\theta_0) \to \varprojlim_{\theta \in P_\sigma} \mathcal{F}(\theta)$$

can be identified with one another. But the condition that $\mathcal{F}$ is ULA over S is equivalent to the requirement that α is an equivalence (for any θ_0 and any τ), while the condition that $\mathcal{F}$ is ULA over X is equivalent to the requirement that β is an equivalence (for any θ_0 and any σ).

To prove $(*)$, suppose we are given an inclusion of simplices $\sigma \subseteq \sigma'$ which belong to Q; we wish to show that the canonical map $\mathcal{G}(\sigma) \to \mathcal{G}(\sigma')$ is an equivalence. Note that the construction $\theta \mapsto \theta \cap f^{-1}\sigma$ induces a map of posets $\lambda : P_{\sigma'} \to P_\sigma$. Condition $(*)$ will follow if we can show that $\mathcal{F}|_{P_\sigma}$ is a right Kan extension of $\mathcal{F}|_{P_{\sigma'}}$ along λ. Unwinding the definitions, this amounts to the assertion that for $\theta \in P_\sigma$, the canonical map

$$\mathcal{F}(\theta) \to \varprojlim_{\theta' \supseteq \theta, f(\theta') = \sigma'} \mathcal{F}(\theta')$$

is an equivalence. This follows from condition (2), applied to any point $s \in S$ which belongs to the interior of the simplex τ. □

Exercise 3. Let $f : X \to S$ be a map of finite polyhedra. Let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathrm{Sp})$ and let $\mathcal{G}$ be a local system of spectra on X. Show that if $\mathcal{F}$ is ULA over S, then so is the constructible sheaf $\mathcal{F} \wedge \mathcal{G}$ given by

$$(\mathcal{F} \wedge \mathcal{G})(\sigma) = \mathcal{F}(\sigma) \wedge \mathcal{G}(\sigma).$$

We now use the theory of ULA sheaves to build a variant of the construction $X \mapsto A(X)$.

Definition 4. Let X be a finite polyhedron. For every finite polyhedron S, we let $\mathrm{Shv}^S_{PL}(X \times S)$ denote the full subcategory of $\mathrm{Shv}_{PL}(X \times S, \mathrm{Sp}^{\mathrm{fin}})$ spanned by those sheaves which are ULA over S. Here $\mathrm{Sp}^{\mathrm{fin}}$ denotes the ∞-category of finite spectra.

Note that the construction $S \mapsto \mathrm{Shv}^S_{PL}(X \times S)$ is contravariant in S (since the condition of being ULA is stable under base change in S). In particular, the construction

$$[n] \mapsto \mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)$$

determines a simplicial object in the ∞-category of stable ∞-categories. Applying the Waldhausen construction pointwise, we obtain a simplicial space

$$K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)).$$

We will denote the geometric realization of this simplicial space by $K_\Delta(X)$.

Remark 5. Recall that each $K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n))$ is the 0th space of a connective spectrum. In the above definition, it does not matter if we take the geometric realization at the level of spaces or at the level of spectra (if we do the latter, we obtain a connective spectrum having $K_\Delta(X)$ as its 0th space).

Our next goal is to relate $K_\Delta(X)$ to the A-theory of X. First, we need a bit of a digression.

Definition 6. Let $\mathcal{C}$ be a stable ∞-category which admits small colimits. We let $\mathcal{C}^\vee$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ spanned by those functors which preserve small colimits. We will refer to $\mathcal{C}^\vee$ as the *dual* of $\mathcal{C}$.

Example 7. Let R be an associative ring spectrum and let $\mathcal{C}$ be the ∞-category of *left* R-modules. Then $\mathcal{C}^\vee$ can be identified with the ∞-category of *right* R-modules; every right R-module M can be identified with the colimit-preserving functor

$$\mathcal{C} \to \mathrm{Sp}$$
$$N \mapsto M \otimes_R N.$$

Example 8. More generally, let $\mathcal{C}$ be a compactly generated stable ∞-category, so that $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the full subcategory spanned by the compact objects. For every object $C \in \mathcal{C}_0$, the functor

$$D \mapsto \mathrm{Map}(C, D)$$

is a colimit-preserving functor from $\mathcal{C}$ to spectra (if we let $\mathrm{Map}(C, D)$ denote the *spectrum* of maps from C to D), which we can identify with an element $h^C \in \mathcal{C}^\vee$. The construction $C \mapsto h^C$ determines a contravariant functor from $\mathcal{C}_0$ to $\mathcal{C}^\vee$, and one can show that this functor induces an equivalence $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \simeq \mathcal{C}^\vee$. In particular, the ∞-category $\mathcal{C}^\vee$ is also compactly generated and its ∞-category of compact objects $\mathcal{C}_0^\vee$ is opposite to the ∞-category $\mathcal{C}_0$. We therefore have a canonical homotopy equivalence of K-theory spaces

$$K(\mathcal{C}_0^\vee) \simeq K(\mathcal{C}_0).$$

Construction 9. Let X be a finite polyhedron and let Sp^X denote the ∞-category of local systems of spectra on X. Let S be another finite polyhedron and let $\phi : X \times S \to X \qquad \psi : X \times S \to S$ be the projection maps. Let $\mathcal{F} \in \mathrm{Shv}_{PL}^S(X \times S)$ and $\mathcal{G} \in \mathrm{Sp}^X$. Then $\mathcal{F} \wedge \phi^* \mathcal{G}$ is a constructible sheaf of spectra on $X \times S$ which is ULA over S (Exercise 3), and therefore $\psi_*(\mathcal{F} \wedge \phi^* \mathcal{G})$ is a local system of spectra on S. This construction yields a map

$$\theta_S : S \to \mathrm{Fun}(\mathrm{Shv}_{PL}^S(X \times S) \times \mathrm{Sp}^X, \mathrm{Sp}).$$

Note that the construction $\mathcal{G} \mapsto \psi_*(\mathcal{F} \wedge \phi^* \mathcal{G})$ preserves colimits in $\mathcal{G}$; we may therefore identify θ_S with a map

$$S \to \mathrm{Fun}(\mathrm{Shv}_{PL}^S(X \times S), (\mathrm{Sp}^X)^\vee).$$

Note that θ_S depends functorially on S.

Let $(\mathrm{Sp}^X)_0^\vee$ denote the full subcategory of $(\mathrm{Sp}^X)^\vee$ spanned by the compact objects (by virtue of Example 8 this is the *opposite* of the ∞-category of compact objects of Sp^X). We claim that θ_S factors through a map

$$S \to \mathrm{Fun}(\mathrm{Shv}_{PL}^S(X \times S), (\mathrm{Sp}^X)_0^\vee).$$

To prove this, it suffices (by functoriality) to treat the case where S is a point; in this case, we are considering a single functor

$$\lambda : \mathrm{Shv}_{PL}(X) \to (\mathrm{Sp}^X)^\vee$$

which is characterized by the formula

$$\lambda(\mathcal{F})(\mathcal{G}) = \psi_*(\mathcal{F} \wedge \mathcal{G}).$$

The collection of those objects $\mathcal{F}$ for which $\lambda(\mathcal{F}) \in (\mathrm{Sp}^X)_0^\vee$ is a stable subcategory of $\mathrm{Shv}_{PL}(X)$. To show that it is all of $\mathrm{Shv}_{PL}(X)$, it will suffice to show that it contains the direct image of any constant sheaf $\underline{S}$ (with value the sphere spectrum) along an inclusion $i : \Delta^k \to X$. If $\mathcal{F} = i_* \underline{S}$, we have

$$\lambda(\mathcal{F})(\mathcal{G}) = \psi_*(i_* \underline{S} \wedge \mathcal{G}) \simeq \mathcal{G}_x$$

for any point $x \in X$ which belongs to the image of i; it follows that the functor $\lambda(\mathcal{F})$ is corepresentable by the (compact) local system $i_{x!}S$, where $i_x : \{x\} \to X$ is the inclusion map.

Passing to K-theory, we see that θ_S induces a map

$$\begin{aligned} K(\theta_S) &\to \mathrm{Fun}(K(\mathrm{Shv}_{PL}^S(X \times S)), K((\mathrm{Sp}^X)_0^\vee)) \\ &\simeq \mathrm{Fun}(K(\mathrm{Shv}_{PL}^S(X \times S)), K((\mathrm{Sp}_0^X)^{\mathrm{op}})) \\ &\simeq \mathrm{Fun}(K(\mathrm{Shv}_{PL}^S(X \times S)), K(\mathrm{Sp}_0^X)) \\ &\simeq \mathrm{Fun}(K(\mathrm{Shv}_{PL}^S(X \times S)), \Omega^\infty A(X)). \end{aligned}$$

These maps depend functorially on S. Specializing to the case where S is a simplex, we obtain a map from the simplicial space $K(\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet))$ to the constant simplicial space with the value $\Omega^\infty A(X)$. Passing to geometric realizations, we obtain a map of infinite loop spaces

$$K_\Delta(X) \to \Omega^\infty A(X).$$

Our goal over the next several lectures is to prove the following:

Theorem 10. *The map* $K_\Delta(X) \to \Omega^\infty A(X)$ *is a model for the assembly map in A-theory. In particular, there is a canonical homotopy equivalence* $K_\Delta(X) \simeq \Omega^\infty(X_+ \wedge A(*))$.