Universal Local Acyclicity (Lecture 23)

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Let $\mathcal{C}$ be an ∞-category, which will remain fixed for most of this lecture In the previous lecture, we introduced the notion of a constructible $\mathcal{C}$-valued sheaf on a finite polyhedron X. Suppose we are given a map of finite polyhedra $f : X \to S$. In this Then a constructible sheaf $\mathcal{F}$ on X determines a family of constructible sheaves $\mathcal{F}_s$ on X_s, as s ranges over the points of S. In this lecture, we will study a condition which guarantees that the sheaves $\mathcal{F}_s$ behave nicely as s varies.

Let us henceforth assume that $\mathcal{C}$ admits finite limits.

Definition 1. Let $f : X \to S$ be a PL map of polyhedra. Choose triangulations $\Sigma(X)$ and $\Sigma(S)$ of X and S which are compatible with the map f. We will say that a sheaf $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ is *universally locally acyclic with respect to* f (ULA) at a simplex $\sigma_0 \in \Sigma(X)$ if the following condition is satisfied:

$(*)$ For any simplex $\tau \in \Sigma(S)$ which contains $\tau_0 = f(\sigma_0)$, the canonical map

$$\mathcal{F}(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma)$$

is an equivalence in $\mathcal{C}$.

We will say that $\mathcal{F}$ is *ULA at a point* $x \in X$ if it is ULA at every simplex which contains x. We will say that $\mathcal{F}$ is ULA if it is ULA at every simplex.

Warning 2. The condition that $\mathcal{F}$ be ULA depends not only on the sheaf $\mathcal{F}$ and the polyhedron X, but also on the map f. When it is important to emphasize this dependence, we will instead use the phrase "$\mathcal{F}$ is ULA over S."

Remark 3. It follows tautologically from the definition that the set of points $x \in X$ at which $\mathcal{F}$ is ULA is open: its complement is a subcomplex of X.

Remark 4. In condition $(*)$, we can replace the poset $\{\sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ with $\{\sigma_0 = \sigma \cap f^{-1}\tau_0, f(\sigma) = \tau\}$: the latter is right cofinal in the former.

Example 5. Let $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$. Then $\mathcal{F}$ is ULA over X if and only if $\mathcal{F}$ is locally constant.

Example 6. Let $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and $f : X \to S$ be as in Definition 1. If $\sigma \in \Sigma(X)$ has the property that $f(\sigma)$ is a maximal simplex of S (not contained as a facet of any larger simplex), then $\mathcal{F}$ is automatically ULA at σ. In particular, $\mathcal{F}$ is automatically ULA on the open subset of X given by the inverse images of the interiors of the maximal simplices of S.

Proposition 7. *The notion of universal local acyclicity is independent of the choice of triangulation. That is, if $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and we choose finer triangulations $\Sigma'(X)$ and $\Sigma'(S)$ (still compatible with the map f) and we let $\mathcal{F}' \in \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ denote the image of $\mathcal{F}$, then $\mathcal{F}$ is ULA at a point $x \in X$ if and only if $\mathcal{F}'$ is ULA at x.*

Proof. Let $g : \Sigma'(X) \to \Sigma(X)$ be the map that takes each simplex of $\Sigma'(X)$ to the smallest simplex of $\Sigma(X)$ which contains it. Then it will suffice to show that $\mathcal{F}'$ is ULA at a simplex σ'_0 if and only if $\mathcal{F}$ is ULA at $\sigma_0 = g(\sigma'_0)$. Suppose first that $\mathcal{F}$ is ULA at σ_0, and consider a simplex $\tau' \in \Sigma'(S)$ which contains $\tau'_0 = f(\sigma'_0)$.

Let τ_0 and τ be the smallest simplices of $\Sigma(S)$ which contain τ'_0 and τ', respectively. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(\sigma_0) & \longrightarrow & \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma)=\tau} \mathcal{F}(\sigma) \\ \downarrow & & \downarrow \\ \mathcal{F}'(\sigma'_0) & \longrightarrow & \varprojlim_{\sigma'_0 \subseteq \sigma', f(\sigma')=\tau'} \mathcal{F}'(\sigma'). \end{array}$$

By hypothesis, the upper horizontal map is an equivalence, and we wish to prove that the same is true of the lower horizontal map. Since the left vertical map is an equivalence, we are reduced to proving the same is true of the lower vertical map. Invoking the definition of $\mathcal{F}'$, we are reduced to proving that g induces a right cofinal map of posets

$$\{\sigma' \in \Sigma'(X) : \sigma'_0 \subseteq \sigma', f(\sigma') = \tau'\} \to \{\sigma \in \Sigma(X) : \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}.$$

Fix a simplex σ in the latter poset; we wish to prove that the poset

$$\{\sigma' \in \Sigma'(X) : \sigma'_0 \subseteq \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$$

is weakly contractible. This follows from the criterion of Lecture 9, applied to the map $\sigma \to \tau$ (with the triangulations induced by $\Sigma'(X)$ and $\Sigma'(S)$); note that this is a linear map of simplices and therefore automatically a fibration.

Now suppose that $\mathcal{F}'$ is ULA at σ'_0; we wish to show that $\mathcal{F}$ is ULA at σ_0. Let τ be any simplex of $\Sigma(S)$ which contains $f(\sigma_0)$. Then we can choose a simplex τ' of $\Sigma'(S)$ which contains $f(\sigma'_0)$ whose interior is contained in the interior of τ. The desired result now follows by applying the above reasoning to the simplex τ'. $\square$

It follows from Proposition 7 that universal local acyclicity makes sense for objects of $\mathrm{Shv}_{PL}(X; \mathcal{C})$. Moreover, it is a condition which is generically satisfied; Example 6 immediately yields the following:

Proposition 8 ("Sard's Theorem")**.** *Let $f : X \to S$ be a map of finite polyhedra. For each $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$, there is a dense open subset $U \subseteq S$ (complementary to a subcomplex of S) such that $\mathcal{F}$ is ULA at every point of $f^{-1}(U)$.*

Example 9. Every sheaf $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$ is ULA with respect to the map $X \to *$.

Proposition 10. *The condition of universal local acyclicity is stable under base change. That is, suppose we are given a pullback square of finite polyhedra*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ S' & \xrightarrow{h} & S \end{array}$$

and a constructible sheaf $\mathcal{F}$ on X. For each point $x' \in X'$, if $\mathcal{F}$ is ULA at $f(x')$, then $f^* \mathcal{F}$ is ULA at x'.

Proof. We repeat the proof of Proposition 7. Choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$ of X and S so that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Similarly, we can choose compatible triangulations of $\Sigma(X')$ and $\Sigma(S')$ such that each simplex of X' maps into a simplex of X (not necessarily by a map which preserves vertices) and similarly for $S' \to S$.

Let σ'_0 be a simplex of $\Sigma(X')$ which contains x', let τ'_0 be its image in S', and let τ' be a simplex of $\Sigma(S')$ which contains τ'_0. We wish to show that the canonical map

$$(f^* \mathcal{F})(\sigma'_0) \to \varprojlim_{\sigma'_0 \subseteq \sigma', g'(\sigma') = \tau'} \mathcal{F}(\sigma')$$

is an equivalence. Let ${\sigma'_0}^+$ denote the smallest element of $\Sigma(X)$ which contains the image of σ'_0, and define ${\tau'_0}^+$ and ${\tau'}^+$ similarly. Then ${\sigma'_0}^+$ contains $f(x')$, so our hypothesis gives an equivalence

$$(f^* \mathcal{F})(\sigma'_0) = \mathcal{F}({\sigma'_0}^+) = \varprojlim_{f(\sigma'_0) \subseteq \sigma, g(\sigma) = {\tau'}^+} \mathcal{F}(\sigma).$$

It will therefore suffice to show that the construction $\sigma' \mapsto {\sigma'}^+$ determines a right cofinal map of posets

$$P = \{\sigma' \in \Sigma(X') : \sigma'_0 \subseteq \sigma', g'(\sigma') = \tau'\} \to Q = \{\sigma \in \Sigma(X) : f(\sigma'_0) \subseteq \sigma, g(\sigma) = {\tau'}^+\}.$$

Fix a simplex $\sigma \in Q$, we wish to show that the poset

$$P_{/\sigma} = \{\sigma' \in \Sigma(X') : \sigma'_0 \subseteq \sigma' \subseteq f^{-1}(\sigma), g'(\sigma') = \tau'\}$$

is weakly contractible. Using the criterion from Lecture 9, we are reduced to showing that the projection map $f^{-1}\sigma \to h^{-1}{\tau'}^+$ is a fibration. Since the diagram in question is a pullback, this reduces to the statement that $\sigma \to {\tau'}^+$ is a fibration, which is clear since it is a linear map of simplices. ☐

Example 11. Let $f : X \to S$ be a map of finite polyhedra. Then the following conditions are equivalent:

(1) The map f is a fibration.

(2) For every ∞-category $\mathcal{C}$ which admits finite limits and every locally constant sheaf $\mathcal{F}$ with values in $\mathcal{C}$, $\mathcal{F}$ is ULA with respect to f.

To prove that (1) $\Rightarrow$ (2), choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$; we wish to verify that a locally constant sheaf $\mathcal{F}$ satisfies condition $(*)$ of Definition 1. That is, we wish to show that certain maps of the form

$$\mathcal{F}(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma)$$

are equivalences in $\mathcal{C}$. Since $\mathcal{F}$ is locally constant, we can identify each $\mathcal{F}(\sigma)$ with a fixed object $C = \mathcal{F}(\sigma_0) \in \mathcal{C}$. The result then follows from the contractibility of the poset $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ (which follows from the combinatorial criterion for f to be a fibration; see Lecture 9).

Conversely, suppose that (2) is satisfied. Taking $\mathcal{C} = \mathcal{S}^{\mathrm{op}}$ and $\mathcal{F}$ to be the constant sheaf taking the value $*$, condition $(*)$ of Definition 1 asserts that posets of the form $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ are weakly contractible, which is equivalent to the requirement that f is a fibration (again by the result of Lecture 9).

Warning 12. We will be primarily interested in the case where $\mathcal{C} = \mathrm{Sp}^{\mathrm{fin}}$ is the ∞-category of finite spectra. In this case, the condition that every locally constant sheaf $\mathcal{F}$ on X be ULA with respect to a map $f : X \to S$ is slightly weaker than the condition that f is a fibration: it implies only that the partially ordered sets $\{\sigma | \sigma_0 \subseteq \sigma, f(\sigma) = \tau\}$ appearing in our fibration criterion are acyclic, not that they are weakly contractible.

Remark 13 (Vanishing Cycles)**.** Let $\mathcal{C}$ be an ∞-category which admits small limits, let $f : X \to S$ be an arbitrary map of topological spaces, and let us consider $\mathcal{C}$-valued sheaves in the sense of Remark **??**.

Suppose we are given a continuous path $p : [0,1] \to S$ with $p(0) = s \in S$. Let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf on X, $\mathcal{F}_p$ denote the restriction of $\mathcal{F}$ to the fiber product $X \times_S [0,1]$, and let $\mathcal{F}_p^\circ$ denote the restriction of $\mathcal{F}_p$ to $X \times_S (0,1]$. Let

$$j : X \times_S (0,1] \to X \times_S [0,1]$$

be the inclusion map; we can then push $\mathcal{F}_p^\circ$ forward to obtain a sheaf $j_* \mathcal{F}_p^\circ$ on $X \times_S [0,1]$. Let $\mathcal{G}$ denote the restriction of $j_* \mathcal{F}_p^\circ$ to the fiber $X \times_S \{0\} = X_s$. We will refer to $\mathcal{G}$ as the *nearby cycles sheaf of $\mathcal{F}$ at the point s in the direction p*. Note that there is a canonical map $\theta : \mathcal{F}|_{X_s} \to \mathcal{G}$. If $\mathcal{C}$ is stable, we will refer the fiber of θ as the *vanishing cycles sheaf of $\mathcal{F}$ at the point s in the direction p*.

In the special case where X and S are finite polyhedra, f is piecewise linear, $\mathcal{F}$ is constructible, then we can choose compatible triangulations $\Sigma(X)$ and $\Sigma(S)$ such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. If $p : [0,1] \to S$ is piecewise linear, then there exist simplices $\tau_0 \subseteq \tau$ of $\Sigma(S)$ such that $s = p(0)$ lies in the interior of τ_0 and $p(t)$ belongs to the interior of τ for sufficiently small nonzero values of t. In this case, there is an identification of the fiber X_s with the nerve of the poset $\{\sigma_0 \in \Sigma(S) : f(\sigma_0) = \tau_0\}$ which exhibits $\mathcal{F}|_{X_s}$ and $\mathcal{G}$ as constructible sheaves on X_s, which correspond to the functors

$$\sigma_0 \mapsto \mathcal{F}(\sigma_0) \qquad \sigma_0 \mapsto \varprojlim_{\sigma_0 \subseteq \sigma, f(\sigma) = \tau} \mathcal{F}(\sigma),$$

respectively. We may therefore reformulate Definition 1 heuristically as follows: a constructible sheaf $\mathcal{F}$ on X is ULA with respect to S if it has no vanishing cycles with respect to any point $s \in S$ and any direction $p : [0,1] \to S$ with $p(0) = s$.

Example 11 shows that if $f : X \to S$ is a fibration, then any locally constant sheaf is universally locally acyclic. However, if we are willing to depart from the world of locally constant sheaves, then the condition of universal local acyclicity has nothing at all to do with the ambient space X:

Proposition 14. *Suppose we are given a commutative diagram of finite polyhedra*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

Let $\mathcal{F} \in \mathrm{Shv}_{PL}(X; \mathcal{C})$ be ULA over S. Then $f_* \mathcal{F} \in \mathrm{Shv}_{PL}(Y)$ is ULA over S. The converse holds if f is a closed embedding.

Proof. Choose compatible triangulations $\Sigma(X)$, $\Sigma(Y)$, and $\Sigma(S)$ such that $\mathcal{F}$ is constructible with respect to $\Sigma(X)$. Fix a simplex $\sigma_0 \in \Sigma(Y)$, let $\tau_0 = g(\sigma_0)$, and let $\tau \in \Sigma(S)$ be a simplex containing τ_0. We wish to show that the canonical map

$$(f_* \mathcal{F})(\sigma_0) \to \varprojlim_{\sigma_0 \subseteq \sigma, g(\sigma) = \tau} (f_* \mathcal{F})(\sigma)$$

is an equivalence. Invoking the definition of f_*, we can write this map as

$$\varprojlim_{f(\theta_0) = \sigma_0} \mathcal{F}(\theta_0) \to \varprojlim_{\sigma_0 \subseteq f(\theta), h(\theta) = \tau} \mathcal{F}(\theta).$$

Let $P = \{\theta_0 \in \Sigma(X) : f(\theta_0) = \sigma_0\}$ and let $Q = \{\theta \in \Sigma(X) : h(\theta) = \tau \text{ and } f(\theta) \supseteq \sigma_0\}$. Intersection with $f^{-1}\sigma_0$ induces a map of posets from Q to P; to complete the proof, it will suffice to show that $\mathcal{F}|_P$ is a right Kan extension of $\mathcal{F}|_Q$ along this map. In other words, it will suffice to show that for each $\theta_0 \in P$, the canonical map

$$\mathcal{F}(\theta_0) \to \varprojlim_{\theta \in Q, \theta_0 \subseteq \theta} \mathcal{F}(\theta)$$

is an equivalence. This follows immediately from our assumption that $\mathcal{F}$ is ULA over S.

Now suppose that f is a closed embedding and that $f_* \mathcal{F}$ is ULA over S. To prove that $\mathcal{F}$ is ULA over S, we must show that for each $\theta_0 \in \Sigma(X)$, if $\tau_0 = h(\theta_0)$ and $\tau \in \Sigma(S)$ contains τ_0, then the canonical map

$$\mathcal{F}(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, h(\theta) = \tau} \mathcal{F}(\theta)$$

is an equivalence. Let us identify X with its image in Y via f, and let us identify $\mathcal{F}$ with the restriction of $f_* \mathcal{F}$ to simplices of X. Then we can write our map as a compsition

$$(g_* \mathcal{F})(\theta_0) \to \varprojlim_{\theta_0 \subseteq \sigma \in \Sigma(Y), g(\sigma)=\tau} (g_* \mathcal{F})(\sigma) \to \varprojlim_{\theta_0 \subseteq \sigma \in \Sigma(X), g(\sigma)=\tau} (g_* \mathcal{F})(\sigma).$$

The first map is an equivalence by virtue of our assumption that $g_* \mathcal{F}$ is ULA over S, and the second map is an equivalence because the restriction of $g_* \mathcal{F}$ to $\{\sigma \in \Sigma(Y) : \theta_0 \subseteq \sigma, g(\sigma) = \tau\}$ is a right Kan extension of the restriction of $\mathcal{F}$ to $\{\sigma \in \Sigma(X) : \theta_0 \subseteq \sigma, g(\sigma) = \tau\}$. □

Remark 15. The proof of Proposition 14 shows something a bit stronger: to verify that $f_* \mathcal{F}$ is ULA over S at a point $y \in Y$, it suffices to know that $\mathcal{F}$ is ULA over S at every point $x \in X$ such that $f(x) = y$.