Constructible Sheaves (Lecture 22)

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For any topological space, one can consider the A-theory assembly map

$$X_+ \wedge A(*) \to A(X)$$

defined in the previous lecture. Our goal over the next few lectures is to provide a "geometric" description of the left hand side, analogous to our description of $A(X)$ as the K-theory of an ∞-category of local systems. In what follows, we will confine our attention to the case where X is a finite polyhedron.

Suppose we are given a PL triangulation of X, which we will identify with a finite partially ordered set $\Sigma(X)$ of simplices in X. In this case, X is homeomorphic to the nerve of the poset $\Sigma(X)$. Consequently, the singular complex $\mathrm{Sing}_\bullet X$ is weakly homotopy equivalent to $\mathrm{N}(\Sigma(X))$. Thinking of both $\mathrm{Sing}_\bullet X$ and $\mathrm{N}(\Sigma(X))$ as ∞-categories, this means that the Kan complex $\mathrm{Sing}_\bullet X$ is obtained from the ∞-category $\mathrm{N}(\Sigma(X))$ by formally inverting all morphisms. In other words, for any ∞-category $\mathcal{C}$, there is a fully faithful embedding

$$\mathcal{C}^X = \mathrm{Fun}(\mathrm{Sing}_\bullet X, \mathcal{C}) \to \mathrm{Fun}(\Sigma(X), \mathcal{C}),$$

whose essential image consists of those functors $\mathcal{F} : \Sigma(X) \to \mathcal{C}$ which carry each inclusion of simplices $\sigma_0 \subseteq \sigma$ to an equivalence $\mathcal{F}(\sigma_0) \to \mathcal{F}(\sigma)$. This motivates the following definition:

Definition 1. Let X be a polyhedron equipped with a triangulation $\Sigma(X)$, and let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued sheaf on X which is constructible with respect to* $\Sigma(X)$ is a functor $\mathrm{N}(\Sigma(X)) \to \mathcal{C}$. We let $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ denote the ∞-category $\mathrm{Fun}(\mathrm{N}(\Sigma(X)), \mathcal{C})$ of $\mathcal{C}$-valued sheaves which are constructible with respect to $\Sigma(X)$.

Example 2. The above analysis shows that we can identify $\mathcal{C}^X$ with a full subcategory of $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$.

Remark 3. There is another notion of sheaf which is less combinatorial in flavor: namely, one can define a *$\mathcal{C}$-valued sheaf on X* (for X any topological space) to be a contravariant functor $\mathcal{F}$ from open subsets of X to $\mathcal{C}$, which satisfies the following descent condition: for any collection of open sets $\{U_\alpha\}$, if we set $U = \bigcup U_\alpha$, then $\mathcal{F}(U) \simeq \varprojlim_V \mathcal{F}(V)$, where the limit is taken over all open subsets $V \subseteq X$ which are contained in some U_α.

If $\Sigma(X)$ is a triangulation of X, one can say that such a sheaf $\mathcal{F}$ is *constructible with respect to* $\Sigma(X)$ if its restriction to the interior of each simplex of $\Sigma(X)$ is a constant sheaf. One can show that if $\mathcal{C}$ admits small limits, then this definition is equivalent to Definition 1. However, since we will only be interested in constructible sheaves, we will be content to work with Definition 1.

Exercise 4. Let Sp denote the ∞-category of spectra. Show that an object $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathrm{Sp})$ is compact if and only if each $\mathcal{F}(\sigma)$ is a finite spectrum.

Suppose that we are given another triangulation $\Sigma'(X)$ which refines a triangulation $\Sigma(X)$ (meaning that each simplex of $\Sigma'(X)$ maps linearly into a simplex of $\Sigma(X)$). Then for each simplex $\sigma' \in \Sigma'(X)$, there is a smallest simplex $\sigma \in \Sigma(X)$ which contains it. The construction $\sigma' \mapsto \sigma$ defines a map of partially ordered sets $f : \Sigma'(X) \to \Sigma(X)$. Composition with f induces a map $\iota : \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$.

Proposition 5. *In the above situation, the functor* $\iota : \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ *is fully faithful.*

Remark 6. Proposition 5 is an immediate consequence of the topological picture described in Remark 3.

Proof. We may assume without loss of generality that $\mathcal{C}$ admits finite limits. The functor ι has a left adjoint $\iota_+ : \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, given by left Kan extension along f. Concretely, this functor can be described by the formula

$$(\iota_+ \mathcal{F})(\sigma) = \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{F}(\sigma'),$$

where $\mathcal{F} \in \mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$. To prove Proposition 5, we must show that for every $\mathcal{G} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, the counit map $\iota_+ \iota^* \mathcal{G} \to \mathcal{G}$ is an equivalence. Evaluating at a simplex $\sigma \in \Sigma(X)$, we are required to prove that $\mathcal{G}(\sigma)$ is given by the colimit $\varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(f(\sigma'))$. In other words, we wish to show that the canonical map

$$\theta : \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(f(\sigma)) \to \varinjlim_{\sigma' \in \Sigma'(X), \sigma' \subseteq \sigma} \mathcal{G}(\sigma)$$

is an equivalence (the right hand side is given by $\mathcal{G}(\sigma)$, since the diagram is indexed by a contractible partially ordered set: in fact, the geometric realization of this partially ordered set is homeomorphic to σ). Let $P_0 = \{\sigma' \in \Sigma'(X) : \sigma' \subseteq \sigma\}$ and let $P_1 = \{\sigma' \in P : f(\sigma') = \sigma\}$. The map θ is determined by a natural transformation between diagrams $S_0 \to \mathcal{C}$, and this natural transformation is invertible when restricted to S_1. To prove that θ is invertible, it suffices to show that S_1 is left cofinal in S_0. This is a special case of the following more general assertion (applied in the case $M = \sigma$):

Lemma 7. *Let M be a piecewise linear n-manifold with boundary, equipped with a triangulation $\Sigma(M)$. Let Q be the collection of simplices of S which are not contained in ∂M. Then the inclusion $Q \hookrightarrow \Sigma(M)$ is left cofinal.*

Remark 8. Lemma 7 can be regarded as an analogue of the assertion that a manifold with boundary is always homotopy equivalent to its interior.

Proof. To prove Lemma 7, we work by induction on n. Fix a simplex $\sigma \in \Sigma(X)$; we wish to show that the set $Q = \{\sigma' \in P : \sigma \subseteq \sigma'\}$ has weakly contractible nerve. If $\sigma \in P$ this is obvious (since the subset above contains σ as a smallest element). Let us therefore assume that σ is a simplex of the boundary ∂M. Let $V = \{\tau \in \Sigma(X) : \sigma \subsetneq \tau\}$. Then V can be identified with the partially ordered set of simplices of $\mathrm{lk}(\sigma)$, which (since M is a PL manifold with boundary) is PL isomorphic to a disk D^m for $m < n$. We can identify Q with the subset of V consisting of simplices which are not contained in ∂D^m. Using the inductive hypothesis, we deduce that the inclusion $Q \hookrightarrow V$ is left cofinal. Since V has weakly contractible nerve, so does Q. □

□

Emboldened by Proposition 5, let us abuse notation by identifying $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ with its essential image in $\mathrm{Shv}_{\Sigma'(X)}(X; \mathcal{C})$ whenever $\Sigma'(X)$ is a refinement of $\Sigma(X)$. This motivates the following:

Definition 9. Let $\mathrm{Shv}_{PL}(X; \mathcal{C})$ denote the filtered direct limit $\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$, where $\Sigma(X)$ ranges over all PL triangulations of X. We will refer to $\mathrm{Shv}_{PL}(X; \mathcal{C})$ as the *∞-category of constructible $\mathcal{C}$-valued sheaves on X.*

Remark 10 (Functoriality)**.** Let $f : X \to Y$ be a PL map of finite polyhedra, and suppose we are given compatible triangulations $\Sigma(X)$ and $\Sigma(Y)$. Then f induces a map of posets $r : \Sigma(X) \to \Sigma(Y)$, which determines a pullback functor

$$f^* : \mathrm{Shv}_{\Sigma(Y)}(Y; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}).$$

If $\mathcal{C}$ admits finite limits, then f^* admits a right adjoint f_*, given by right Kan extension along r. Since r is a Cartesian fibration, this right Kan extension can be described concretely by the formula

$$(f_* \mathcal{F})(\tau) = \varprojlim_{f(\sigma) = \tau} \mathcal{F}(\sigma).$$

Suppose that we are given refinements $\Sigma'(X)$ and $\Sigma'(Y)$ of the triangulations $\Sigma(X)$ and $\Sigma(Y)$, which are compatible with the map f. It is easy to see that the definition of f^* is compatible with the fully faithful embeddings of Proposition 5. We claim that the same is true of f_*. In other words, we claim that if $\mathcal{F} \in \mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C})$ and $\tau' \in \Sigma'(Y)$, then the canonical map

$$\varprojlim_{f(\sigma)={\tau'}^{+}} \mathcal{F}(\sigma) \to \varprojlim_{f(\sigma')=\tau'} \mathcal{F}((\sigma')^{+})$$

is an equivalence, where ${\tau'}^{+}$ Is the smallest simplex of $\Sigma(Y)$ containing τ' and the notation $(\sigma')^{+}$ is defined similarly. To prove this, it suffices to show that construction $\sigma' \mapsto (\sigma')^{+}$ defines a right cofinal map of posets

$$\{\sigma' \in \Sigma'(X) : f(\sigma') = \tau'\} \to \{\sigma \in \Sigma(X) : f(\sigma) = (\tau')^{+}\}.$$

Fix a simplex $\sigma \in \Sigma(X)$ with $f(\sigma) = ({\tau'}^{+})$; we wish to prove that the poset $\{\sigma' \in \Sigma'(X) : \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$ is weakly contractible. This follows by applying our criterion for cell-like maps to the map of simplices $\sigma \to (\tau')^{+}$ (equipped with the triangulations induced by $\Sigma'(X)$ and $\Sigma'(Y)$, respectively).

Passing to the limit over all triangulations, we obtain a pair of adjoint functors

$$\mathrm{Shv}_{PL}(Y; \mathcal{C}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Shv}_{PL}(X; \mathcal{C}).$$

If $\mathcal{C}$ admits finite colimits, then the functor f^* also admits a left adjoint

$$\mathrm{Shv}_{\Sigma(X)}(X; \mathcal{C}) \to \mathrm{Shv}_{\Sigma(Y)}(Y; \mathcal{C}).$$

This functor is *not* compatible with refinement of triangulation, and will therefore be of little use to us.