The Algebraic K-Theory of Spaces (Lecture 21)

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Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex, and in particular an ∞-category. If $\mathcal{C}$ is another ∞-category, we define a *local system on X with values in* $\mathcal{C}$ to be a map of simplicial sets

$$\mathrm{Sing}_\bullet(X) \to \mathcal{C}.$$

The collection of all local systems on X with values in $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Fun}(\mathrm{Sing}_\bullet(X), \mathcal{C})$, which we will denote by $\mathcal{C}^X$.

Example 1. If $\mathcal{C}$ is an ordinary category, then every local system on X with values in $\mathcal{C}$ factors through the homotopy category of $\mathrm{Sing}_\bullet(X)$, which is the *fundamental groupoid* of X. If X is connected and we choose a base point $x \in X$, then we can identify $\mathcal{C}^X$ with the category consisting of objects $C \in \mathcal{C}$ with an action of the fundamental group $\pi_1(X, x)$.

Variant 2. We will generally use the term "space" to refer either to a topological space or to a Kan complex (or to an object of some other type which could be used as a model for homotopy theory). In the latter case, the notion of local system takes a simpler form: it is just a map from X into $\mathcal{C}$.

In what follows, we will confine our attention to the case where $\mathcal{C}$ is the ∞-category Sp of spectra. In this case, we will refer to objects of Sp^X as *local systems of spectra on X* or *spectra parametrized by X*. However, many of the notions we introduce make sense for more general ∞-categories $\mathcal{C}$.

Notation 3. Let X be a space and let $\mathcal{L}$ be a local system of spectra on X. Then each point $x \in X$ determines a spectrum $\mathcal{L}_x$, which we will refer to as the *value of $\mathcal{L}$ at x*.

Remark 4. The ∞-category Sp admits small limits and colimits. Consequently, given a local system $\mathcal{L}$ of spectra on a space X, we can take its limit or colimit to obtain a spectrum. We will denote the limit by $C^*(X; \mathcal{L})$ and the colimit by $C_*(X; \mathcal{L})$. In the special case where the local system $\mathcal{L}$ is constant with value E, these can be identified with the function spectrum E^X and the smash product $E \wedge X_+$, respectively.

Remark 5 (Functoriality)**.** Let $f : X \to Y$ be a map of spaces. Then composition with f determines a pullback functor $f^* : \mathrm{Sp}^Y \to \mathrm{Sp}^X$. We will sometimes denote the pullback of a local system $\mathcal{L} \in \mathrm{Sp}^X$ by $\mathcal{L}\,|_Y$.

It follows from abstract nonsense that the functor f^* admits both a left adjoint $f_!$ and a right adjoint f_* (given by left and right Kan extension). If f is a fibration (which we can always arrange), then these functors are given by the formula

$$(f_!\,\mathcal{F})_y = C_*(X_y; \mathcal{L}\,|_{X_y}) \qquad (f_*\,\mathcal{F})_y = C^*(X_y; \mathcal{L}\,|_{X_y})$$

where X_y denotes the fiber of f over the point y.

Proposition 6. *Let X be a space. Then the ∞-category Sp^X is compactly generated. That is, it is equivalent to $\mathrm{Ind}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathrm{Sp}^X$ is the full subcategory spanned by the compact objects.*

Proof. It follows from general nonsense that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Sp}^X$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{Sp}^X$, and that F admits a right adjoint G. To show that F is an equivalence of ∞-categories, it suffices to show that G is conservative. In other words, it will suffice to show that if $\alpha : \mathcal{L} \to \mathcal{L}'$ is a morphism of local systems and that $G(\alpha)$ is an equivalence, then α is an equivalence. Pick a point $x \in X$, and let $i : \{x\} \hookrightarrow X$ denote the inclusion map. The functor i^* preserves filtered colimits, so its left adjoint $i_!$ preserves compact objects. If $G(\alpha)$ is an equivalence, we conclude that it induces a homotopy equivalence

$$\mathrm{Map}(i_! E, \mathcal{L}) \to \mathrm{Map}(i_! E, \mathcal{L}')$$

for every finite spectrum E (viewed as a local system on $\{x\}$). It follows that the map $\mathrm{Map}(E, \mathcal{L}_x) \to \mathrm{Map}(E, \mathcal{L}'_x)$ is a homotopy equivalence for every finite spectrum E, from which we conclude that $\mathcal{L}_x \simeq \mathcal{L}'_x$. Since x is arbitrary, it follows that α is an equivalence. □

The proof of Proposition 6 shows something a bit stronger: the ∞-category Sp^X is generated (under colimits and desuspensions) by compact objects of the form $i_! S$, where S is the sphere spectrum and i ranges over the inclusions of all points$x \in X$. It follows that the collection of compact objects of Sp^X is generated (under finite colimits, desuspensions, and retracts) by objects of the form $i_! S$. Moreover, it suffices to consider one point x lying in each connected component of X. Consequently, if X is connected, then Sp^X is generated by a single compact object $i_! S$, and is therefore equivalent to the ∞-category Mod_R where $R = \mathrm{End}(i_! S)$ is the ring spectrum of endomorphisms of $i_! S$. Note that we can identify R with the spectrum of maps from S to $i^* i_! S$: that is, with the value of $i_! S$ at the point x. Converting i into a fibration and using Remark 5, we see that R can be identified with the spectrum

$$C_*(\Omega(X); S) \simeq \Sigma^\infty_+ \Omega(X).$$

Note that R is connective ring spectrum and that $\pi_0 R$ is isomorphic to the group algebra $\mathbf{Z}[\pi_1 X]$ (specializing to the case of *discrete* R-modules, we recover a more familiar fact: the category of local systems of abelian groups on X is equivalent to the category of $\mathbf{Z}[\pi_1 X]$-modules).

Definition 7. Let X be a space and let $\mathcal{C} \subseteq \mathrm{Sp}^X$ be the full subcategory spanned by the compact objects. Then $K(\mathcal{C})$ is a grouplike E_∞-space, and is therefore the 0th space of a connective spectrum. We will denote this spectrum by $A(X)$ and refer to it as the *A-theory spectrum of* X.

In what follows, we will generally abuse notation and not distinguish between grouplike E_∞-spaces and the corresponding spectra.

Example 8. Let X be a connected space with base point $x \in X$. Then we have $A(X) \simeq K(R)$, where $R = \Sigma^\infty_+ \Omega X$ is the ring spectrum described above. Since R is connective, we can identify $A(X)$ with the group completion of the E_∞-space $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ of finitely generated projective R-modules.

Warning 9. Our definition of $A(X)$ is not standard. The usual convention in the literature is to use $A(X)$ to refer to the group completion of the E_∞-space of finitely generated *free* R-modules. However, we have seen that it does not make a very big difference: the only thing that changes is the group $\pi_0 A(X)$.

Remark 10. Let X be a connected space and let G be its fundamental group. Applying the results of the previous lecture, we obtain isomorphisms

$$\pi_0 A(X) = K_0(\mathbf{Z}[G]) \qquad \pi_1 A(X) = K_1(\mathbf{Z}[G]) = \mathrm{GL}_\infty(\mathbf{Z}[G])^{\mathrm{ab}}.$$

However, the higher homotopy groups of $A(X)$ do not have "classical" names: they depend on the entire homotopy type of X (rather than just its fundamental group) and on the fact that we are working over the sphere spectrum (rather than the ring $\mathbf{Z}$ of integers).

Example 11. Let X be a simply connected space. Then we have

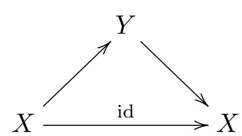
$$\pi_0 A(X) \simeq \mathbf{Z} \qquad \pi_1 A(X) \simeq \mathbf{Z}/2\mathbf{Z}.$$

Let $f : X \to Y$ be a map of spaces. Since the pullback functor f^* preserves filtered colimits, its left adjoint $f_!$ preserves compact objects and therefore induces a map of K-theory spectra $A(X) \to A(Y)$. Consequently, we can view the construction $X \mapsto A(X)$ as a covariant functor from the ∞-category of spaces to the ∞-category of spectra.

Let us now consider some other types of local system:

Example 12. Let $\mathcal{S}$ denote the ∞-category of spaces. For every space X, we can identify the ∞-category $\mathcal{S}^X$ of local systems on X with the ∞-category $\mathcal{S}_{/X}$ of spaces Y with a map $Y \to X$; the identification associates to each map $f : Y \to X$ the local system $x \mapsto Y_x$ where Y_x denotes the homotopy fiber of f over the point $x \in X$. The proof of Proposition 6 shows that $\mathcal{S}^X$ is compactly generated. Under the equivalence $\mathcal{S}^X \simeq \mathcal{S}_{/X}$, the compact objects correspond to those maps $Y \to X$ where Y is a finitely dominated space (note that the finiteness condition here is placed on the space Y itself, not on the homotopy fibers of the map $Y \to X$).

Example 13. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces. Then the identification $\mathcal{S}^X \simeq \mathcal{S}_{/X}$ of Example 12 induces an identification $\mathcal{S}_*^X \simeq \mathcal{S}_{X//X}$, where $\mathcal{S}_{X//X}$ denotes the ∞-category of diagrams



which exhibit X as a retract of Y. The proof of Proposition 6 shows that this ∞-category is generated by compact objects. Examples of compact objects include any diagram as above where Y can be obtained from X by attaching finitely many cells. Conversely, any compact object is a *retract* (in the homotopy category) of such a relative cell complex.

There are evident maps

$$\mathcal{S}^X \to \mathcal{S}_*^X \to \mathrm{Sp}^X,$$

given pointwise by "adding a disjoint basepoint" and "taking the suspension spectrum." These constructions preserve compact objects (since they are left adjoint to functors which preserve filtered colimits). In particular, if Y is a finitely dominated space over X, then the construction $x \mapsto \Sigma^\infty_+(Y_x)$ determines a compact object of Sp^X, which determines a point of the space $\Omega^\infty A(X)$ which we will denote by $[Y]$.

Example 14 (Wall Finiteness Obstruction)**.** Suppose that the space X itself is finitely dominated. Then the above construction determines a point $[X] \in \Omega^\infty A(X)$, which is represented by the constant local system $\underline{S}$ which takes each point of x to the sphere spectrum S. We let $\overline{w}_X \in \pi_0 A(X)$ denote the class represented by this point.

Suppose that X is connected with fundamental group G. We claim that under the isomorphism $\pi_0 A(X) \simeq K_0(\mathbf{Z}[G])$ of Remark 10, the class $\overline{w}_X$ is a *lifting* of the Wall finiteness obstruction $w_X \in \widetilde{K}_0(\mathbf{Z}[G])$ introduced in Lecture 2. Recall that to define w_X, we chose a finite complex X' with a map $X' \to X$ such that the relative homology $\mathrm{H}_*(X, X'; \mathbf{Z}[G])$ was a projective module P concentrated in a single degree n, and defined $w_X = (-1)^n[P]$. Choose a base point $x \in X$ and set $R = \Sigma^\infty_+ \Omega(X)$, so that every map of spaces $Y \to X$ determines an R-module spectrum $\Sigma^\infty_+ Y_x$. We then have a cofiber sequence of R-modules

$$\Sigma^\infty_+ X'_x \to \Sigma^\infty_+ X_x \to \Sigma^n \overline{P},$$

where $\overline{P}$ is a projective R-module with $\pi_0 \overline{P} = P$. Since X' admits a finite cell decomposition, the R-module spectrum $\Sigma^\infty_+ X'_x$ admits a finite filtration whose successive quotients are suspensions of R and therefore represents a class in $K(R)$ given by some integer m. We then have

$$\overline{w}_X = m + (-1)^n[P]$$

in $\pi_0 A(X) \simeq K_0(R) \simeq K_0(\mathbf{Z}[G])$.

The abstract version of the Wall finiteness criterion given in Lecture 15 asserts that a finitely dominated space X is homotopy equivalent to a finite cell complex if and only if $\overline{w}_X$ belongs to the image of the canonical map $K_0(\mathcal{S}^{\text{fin}}_{X//X}) \to \pi_0 A(X)$, where $\mathcal{S}^{\text{fin}}_{X//X}$ is the full subcategory of $\mathcal{S}_{X//X}$ spanned by the finite relative cell complexes. It is not hard to see (and we have already invoked above) that the the image of this map is precisely the subgroup $\mathbf{Z} \subseteq K_0(\mathbf{Z}[G])$ corresponding to projective $\mathbf{Z}[G]$-modules which are *free*. We therefore obtain an alternative proof of the main result of lecture 2: the space X is finitely dominated if and only if w_X vanishes in $K_0(\mathbf{Z}[G])$.

Remark 15 (Assembly Maps)**.** Let $\mathcal{S}$ denote the ∞-category of spaces and let $\mathcal{C} \subseteq \mathcal{S}$ be the full subcategory consisting only of the 1-point space $*$. For any functor $F : \mathcal{S} \to \mathrm{Sp}$, we can identify the restriction $F|_{\mathcal{C}}$ with a single spectrum $F(*)$. Let F_+ be the left Kan extension of $F|_{\mathcal{C}}$ along the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}$: this is the functor given by

$$F_+(X) = \varinjlim_{C \to X} F(C)$$

where C ranges over all objects of $\mathcal{C}$ equipped with a $f : C \to X$. By definition, we must have $C = *$ and we can identify f with a point $x \in X$, so that $F_+(X)$ can be identified with the spectrum $C_*(X; F(*)) = X_+ \wedge F(*)$.

The universal property of the left Kan extension F_+ guarantees that there is a natural transformation of functors $F_+ \to F$, determined uniquely (up to homotopy) by the requirement that it is the identity map when evaluated at a point. In other words, for any space X we have a canonical map

$$C_*(X; F(*)) \to F(X).$$

We will refer to this map as *the assembly map* associated to F. It is an equivalence if and only if the functor F commutes with small colimits (in which case F is determined by the spectrum $F(*)$).

Specializing Remark 15 to the case where F is the A-theory functor $X \mapsto A(X)$, we obtain the A-theory assembly map

$$C_*(X; A(*)) \to A(X).$$

This map is *not* an equivalence in general, and we will see that its failure to be an equivalence measures the difference between simple homotopy theory and ordinary homotopy theory.

Definition 16. For every space X, we let $\mathrm{Wh}(X)$ denote the cofiber of the assembly map $C_*(X; A(*)) \to A(X)$. We will refer to $\mathrm{Wh}(X)$ as the *(piecewise linear) Whitehead spectrum* of X.

Remark 17. Let X be a connected space with fundamental group G. Using the isomorphisms $\pi_0 A(*) \simeq \mathbf{Z}$ and $\pi_1 A(*) \simeq \mathbf{Z}/2\mathbf{Z}$, the Atiyah-Hirzebruch spectral sequence supplies an isomorphism

$$\mathrm{H}_0(X; A(*)) \simeq \mathrm{H}_0(X; \mathbf{Z}) \simeq \mathbf{Z}$$

and an exact sequence of low-degree terms

$$\mathrm{H}_0(X; \pi_1 A(*)) \to \mathrm{H}_1(X; A(*)) \to \mathrm{H}_1(X; \mathbf{Z}) \to 0.$$

This sequence is exact on the left and canonically split (we can see this by considering the projection map from X to a point), so we obtain an isomorphism

$$\mathrm{H}_1(X; A(*)) \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus G^{\mathrm{ab}}.$$

The cofiber sequence of spectra

$$C_*(X; A(*)) \to A(X) \to \mathrm{Wh}(X)$$

now supplies a long exact sequence of abelian groups

$$(\mathbf{Z}/2\mathbf{Z}) \oplus G^{\mathrm{ab}} \xrightarrow{\beta} K_1(\mathbf{Z}[G]) \to \pi_1 \operatorname{Wh}(X) \to \mathbf{Z} \xrightarrow{\alpha} K_0(\mathbf{Z}[G]) \to \pi_0 \operatorname{Wh}(X) \to 0.$$

The map α is split injective (via the ring homomorphism $\mathbf{Z}[G] \to \mathbf{Z}$ which annihilates G, say). We can therefore identify $\pi_0 \operatorname{Wh}(X)$ with the reduced K-group $\widetilde{K}_0(\mathbf{Z}[G])$ and $\pi_1 \operatorname{Wh}(X)$ with the cokernel of β, which is the Whitehead group of X as defined in Lecture 4.

For our applications, it will be convenient to have a *geometric* understanding of the assembly map: that is, we would like to understand it not as arising from the general categorical construction of Remark 15, but instead have an interpretation of the domain $C_*(X; A(*))$ as related to some sorts of kind of sheaf theory on X, just as $A(X)$ is related to local systems on X. We will take this up in the next lecture.