Lower K-Groups of Ring Spectra (Lecture 20)

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We begin by finishing the proof from the previous lecture. Recall that R is a connective associative ring spectrum, that $\mathrm{Mod}_R^{(n)}$ denotes the ∞-category of connective perfect R-modules which have projective amplitude $\leq n$, and that $\mathcal{C}$ is the ∞-category of cofiber sequences

$$M' \to M \to M''$$

where $M' \in \mathrm{Mod}_R^{(n-1)}$, $M \in \mathrm{Mod}_R^{(0)} = \mathrm{Mod}_R^{\mathrm{proj}}$, and $M'' \in \mathrm{Mod}_R^{(n)}$. We regard $S_\bullet \mathrm{Mod}_R^{\mathrm{proj}}$ as a simplicial E_∞-space (with E_∞-structure given by the formation of direct sums), which acts on $S_\bullet \mathcal{C}$ via the construction

$$(P, M' \to M \to M'') \mapsto (M' \oplus P \to M \oplus P \to M'').$$

We wish to prove the following:

Proposition 1. *The canonical map $S_\bullet \mathcal{C} / S_\bullet \mathrm{Mod}_R^{\mathrm{proj}} \to S_\bullet \mathrm{Mod}_R^{(n)}$ is a homotopy equivalence of simplicial spaces.*

Proposition 1 is a levelwise statement: it asserts that for each integer m, the bar construction

$$S_m \mathcal{C} / S_m \mathrm{Mod}_R^{\mathrm{proj}}$$

is homotopy equivalent to $S_m \mathrm{Mod}_R^{(n)}$. To simplify the exposition, we will consider only the case $m = 1$ (the general case follows by essentially the same argument). In this case, we wish to prove that the natural map

$$\theta : \mathcal{C}^{\simeq} / (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to (\mathrm{Mod}_R^{(n)})^{\simeq}$$

is a homotopy equivalence. Fix an object $M \in \mathrm{Mod}_R^{(n)}$; we wish to show that the homotopy fiber $\theta^{-1}\{M\}$ is contractible. Set $\mathcal{C}_M = \mathcal{C} \times_{\mathrm{Mod}_R^{(n)}} \{M\}$, so that we can identify $\mathcal{C}_M$ with the ∞-category whose objects are finitely projective R-modules P equipped with a map $P \to M$ which is surjective on π_0. Then $\theta^{-1}\{M\}$ can be identified with the bar construction $\mathcal{C}_M^{\simeq} / (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$, where $\mathrm{Mod}_R^{\mathrm{proj}}$ acts on $\mathcal{C}_M^{\simeq}$ by the construction

$$(Q, \lambda : P \to M) \mapsto ((\lambda \oplus 0) : P \oplus Q \to M).$$

We wish to show that this space is contractible.

Let us first consider the special case where M is projective. In this case, any map $\lambda : P \to M$ which is surjective on π_0 is automatically split, so that we can write P as a direct sum $M \oplus P_0$ where λ is the identity on M and vanishes on P_0. Here P_0 is a direct summand of P and therefore also projective.

If $P \to M$ and $Q \to M$ are both surjective on π_0, we deduce that $P \times_M Q$ has a direct sum decomposition $M \oplus P_0 \oplus Q_0$, where P_0 and Q_0 are direct summands of P and Q. Consequently, if P and Q are both projective, then so is $P \times_M Q$. It follows that the ∞-category $\mathcal{C}_M$ admits finite products, so that $\mathcal{C}_M^{\simeq}$ has the structure of an E_∞-space. We can now proceed as in Lecture 18. The action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{C}_M^{\simeq}$ is via an E_∞-map

$$P \mapsto (P \oplus M \to M)$$

which is surjective on connected components. It follows that the quotient $(\mathcal{C}_M^{\simeq})/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ is connected and in particular grouplike. Consequently, to show that it is contractible, it will suffice to show that the group completion $(\mathcal{C}_M^{\simeq})/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq})^{\mathrm{grp}}$ is contractible. Equivalently, we must show that the map

$$f : (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to \mathcal{C}_M^{\simeq}$$

induces a homotopy equivalence after group completion. Note that f has a left homotopy inverse q, given by

$$(\lambda : P \to M) \mapsto \mathrm{fib}(\lambda).$$

To complete the proof, it will suffice to show that q is also a right homotopy inverse after group completion. The composition $f \circ q$ is given by

$$(\lambda : P \to M) \to (\mathrm{fib}(\lambda) \oplus M \to M).$$

This is not homotopic to the identity functor from $\mathcal{C}_M^{\simeq}$ to itself. However, we claim that $(f \circ q) + \mathrm{id}$ is homotopic to $\mathrm{id} \oplus \mathrm{id}$: that is, for each object $(\lambda : P \to M)$, there is a canonical equivalence

$$P \times_M P \simeq P \oplus \mathrm{fib}(\lambda)$$

(which is compatible with the projection to M). This follows from the fact that either projection map $P \times_M P \to P$ has a canonical section, given by the diagonal map $\delta : P \to P \times_M P$. This completes the proof in the case where M is projective.

We now consider the general case. Let $\mathcal{D} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}_M)$ be the subcategory whose objects are diagrams

$$P \xrightarrow{\alpha} Q \xrightarrow{\beta} M$$

where Q is projective and the maps α and β are surjective on π_0, and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & M \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \mathrm{id}} \\ P' & \longrightarrow & Q' & \longrightarrow & M \end{array}$$

where γ is an equivalence and δ is surjective on π_0. There is an obvious forgetful functor $\mathcal{D} \to \mathcal{C}_M^{\simeq}$ given by "forgetting" the middle term. This functor is a fibration whose fibers are weakly contractible (they have initial objects, given by taking $Q = P$), and therefore a weak homotopy equivalence. Note that the action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{C}_M^{\simeq}$ extends to an action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ on $\mathcal{D}$, given by

$$(P', P \to Q \to M) \mapsto (P \oplus P' \to Q \to M).$$

It will therefore suffice to show that the quotient $\mathcal{D}/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ is weakly contractible.

Let $\mathcal{C}_M^{\mathrm{surj}}$ denote the subcategory of $\mathcal{C}_M$ containing all objects, whose morphisms are given by commutative diagrams

$$\begin{array}{ccc} Q & \longrightarrow & Q' \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

where the morphisms are surjective on π_0. There is an evident forgetful functor $\mathcal{D} \to \mathcal{C}_M^{\mathrm{surj}}$ (given by forgetting P) which is equivariant with respect to the action of $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ (where we regard $(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$ as acting trivially on $\mathcal{C}_M^{\mathrm{surj}}$), and therefore induces a map

$$\psi : \mathcal{D}/(\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq} \to \mathcal{C}_M^{\mathrm{surj}}.$$

This map is a left fibration whose fiber over an object $(Q \to M)$ can be identified with the bar construction $\mathcal{C}^{\simeq}_{Q} / \operatorname{Mod}^{\mathrm{proj}}_{R}$, which we have already proved to be contractible. It follows that ψ is an equivalence of ∞-categories, and therefore a weak homotopy equivalence. We are now reduced to proving the following:

Lemma 2. *The ∞-category $\mathcal{C}^{\mathrm{surj}}_{M}$ is weakly contractible.*

Proof. Note that $\mathcal{C}^{\mathrm{surj}}_{M}$ is equipped with a multiplication, given by

$$((\lambda : Q \to M), (\lambda' : Q' \to M)) \to (\lambda \oplus \lambda' : Q \oplus Q' \to M).$$

Let X denote the topological space $|\,\mathcal{C}^{\mathrm{surj}}_{M}\,|$, so that we obtain a multiplication map $m : X \times X \to X$. The composite map

$$X \to X \times X \stackrel{m}{\to} X$$

is canonically homotopic to the identity; this follows from the existence of a canonical commutative diagram

$$\begin{array}{ccc} Q \oplus Q & \xrightarrow{\lambda \oplus \lambda} & M \\ \downarrow & & \downarrow \mathrm{id} \\ Q & \xrightarrow{\lambda} & M \end{array}$$

where the left vertical map is given by the fold. In particular, for any $x \in X$ we obtain a canonical path h_x from $m(x,x)$ to x. Let ψ denote the induced map

$$\pi_n(X,x) \times \pi_n(X,x) \stackrel{m}{\to} \pi_n(X, m(x,x)) \stackrel{h_x}{\to} \pi_n(X,x).$$

Then ψ is a group homomorphism, hence given by

$$\psi(a,b) = \phi_1(a)\phi_2(b)$$

for some commuting group homomorphisms $\phi_1, \phi_2 : \pi_n(X,x) \to \pi_n(X,x)$. By symmetry we must have $\phi_1 = \phi_2$; let us denote them both by ϕ. Using the fact that h extends to a homotopy of the composite map

$$X \to X \times X \stackrel{m}{\to} X$$

to the identity, we see that $a = \psi(a,a) = \phi(a)^2 = \phi(a^2)$ for all $a \in \pi_n(X,x)$. In particular, the map ϕ is surjective. Using the associativity of the direct sum (and its compatibility with the construction of the homotopy h), we also deduce the identity

$$\psi(a, \psi(b,c)) = \psi(\psi(a,b), c)$$

$$\phi(a)\phi(\phi(b))\phi(\phi(c)) = \phi(\phi(a))\phi(\phi(b))\phi(c)$$

Taking c to be the identity and cancelling, we obtain $\phi(a) = \phi(\phi(a))$: that is, ϕ is the identity when restricted to $\operatorname{Im}(\phi)$. Since ϕ is surjective, we conclude that $\phi = \mathrm{id}$. Thus $a = a^2$ for all $a \in \pi_n(X,x)$, which shows that the connected component of x in X is contractible.

To complete the proof, it will suffice to show that X is connected. This follows from the observation that for any maps $\lambda : Q \to X$ and $\lambda' : Q' \to X$ which are surjective on π_0, we can find a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \lambda \\ Q' & \xrightarrow{\lambda} & X \end{array}$$

where P is finitely generated and projective and all maps are surjective on π_0. To see this, we note that the fiber product $Q \times_X Q'$ is connective and perfect and therefore there exists map $R^n \to Q \times_X Q'$ which is surjective on π_0. $\square$

This completes the proof that for a connective associative ring spectrum R, we have a homotopy equivalence $K(R) \simeq K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$.

Corollary 3. *Let R be a connective associative ring spectrum. Then there is a canonical isomorphism $\pi_0 K(R) \simeq K_0(\pi_0 R)$, where K_0 is defined as in Lecture 2 (so that $K_0(A)$ denotes the Grothendieck group of finitely generated projective A-modules).*

Proof. We have $\pi_0 K(R) = \pi_0 K_{\mathrm{add}}(R) = \pi_0(\mathrm{Mod}_R^{\mathrm{proj}})^{\mathrm{grp}}$, so that $\pi_0 K(R)$ can be identified with the Grothendieck group of the commutative monoid of equivalence classes of finitely generated projective R-modules. To complete the proof, it will suffice to establish the following:

- $(*)$ Let R be a connective associative ring spectrum. Then the construction $P \mapsto \pi_0 P$ induces an equivalence from the homotopy category of (finitely generated) projective R-modules to the ordinary category of (finitely generated) projective $\pi_0 R$-modules.

Note that if P is projective, then we have a canonical isomorphism

$$\pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(P, M) \simeq \mathrm{Hom}_{\pi_0 R}(\pi_0 P, \pi_0 M)$$

for *every* R-module spectrum M (observe that that the collection of all R-modules P for which this map is bijective is closed under retracts, direct sums, and contains R). This proves that the functor described in $(*)$ is fully faithful. To prove essential surjectivity, let P_0 be a finitely generated projective module over $\pi_0 R$, so that P_0 is the image of some idempotent map $e_0 : (\pi_0 R)^n \to (\pi_0 R)^n$. Using the full faithfulness, we can lift e_0 to a map $e : R^n \to R^n$. Let P be the direct limit

$$R^n \xrightarrow{e} R^n \xrightarrow{e} \cdots$$

and let Q be the direct limit

$$R^n \xrightarrow{1-e} R^n \xrightarrow{1-e} \cdots .$$

By computing homotopy groups, we see that the natural maps $R^n \to P$ and $R^n \to Q$ exhibit R^n as a product of P with Q. In particular, P is a finitely generated projective R-module, and we clearly have $\pi_0 P = P_0$. □

Corollary 4. *Let R be a connective associative ring spectrum. Then there is a canonical isomorphism $\pi_1 K(R) \simeq K_1(\pi_0 R)$, where K_1 is defined as in Lecture 3 (that is, we have $\pi_1 K(R) \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}$).*

Proof. Let $X = (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$, so that $K(R)$ can be identified with (the 0th space of) the group completion of X. In other words, $K(R)$ is universal among E_∞-spaces which receive a map $X \to K(R)$ having the property that every element of $\pi_0 X$ becomes invertible in $\pi_0 K(R)$. We can identify the elements of $\pi_0 X$ with the equivalence classes of finitely generated projective R-modules. Each of these appears as a direct summand of R^n; consequently, $K(R)$ is universal among E_∞-spaces which receive a map from X for which the image of the point $\{R\}$ becomes invertible in $\pi_0 K(R)$. At the level of singular chain complexes, it follows that $C_*(K(R); \mathbf{Z})$ is universal among E_∞-algebras over $\mathbf{Z}$ which receive an E_∞-map $C_*(X; \mathbf{Z})$ for which the point $[R] \in \mathrm{H}_0(X; \mathbf{Z})$ becomes invertible in $\mathrm{H}_0(K(R); \mathbf{Z})$: that is, it can be identified with the E_∞-algebra obtained from $C_*(X; \mathbf{Z})$ by inverting $[R]$. In particular, we can identify $\mathrm{H}_1(K(R); \mathbf{Z})$ with the direct limit of the sequence

$$\mathrm{H}_1(X; \mathbf{Z}) \xrightarrow{[R]} \mathrm{H}_1(X; \mathbf{Z}) \xrightarrow{[R]} \mathrm{H}_1(X; \mathbf{Z}) \to \cdots .$$

Restricting our attention to the identity component $K(R)^\circ \subseteq K(R)$, we can identify $\mathrm{H}_1(K(R)^\circ; \mathbf{Z})$ with the direct limit

$$\mathrm{H}_1(X(0); \mathbf{Z}) \to \mathrm{H}_1(X(1); \mathbf{Z}) \to \mathrm{H}_1(X(2); \mathbf{Z}) \to \cdots ,$$

where each $X(n)$ is the connected component of X which classifies R-modules which are equivalent to R^n (and the transition maps are given by forming the direct sum with a copy of R). In particular, each $X(n)$

is a classifying space for the group $\mathrm{Aut}(R^n)$ of automorphisms of R^n (in the ∞-category Mod_R). We then have $\pi_1 X(n) = \pi_0 \mathrm{Aut}(R^n) \simeq \mathrm{GL}_n(\pi_0 R)$, so that

$$\pi_1 K(R) \simeq \mathrm{H}_1(K(R)^\circ; \mathbf{Z}) \simeq \varinjlim \mathrm{H}_1(X(n); \mathbf{Z}) \simeq \varinjlim (\pi_1 X(n))^{\mathrm{ab}} \simeq \varinjlim \mathrm{GL}_n(\pi_0 R)^{\mathrm{ab}} \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}.$$

□

Warning 5. Corollaries 3 and 4 show that the groups $K_0(R)$ and $K_1(R)$ of a connective ring spectrum R depend only on the associative ring $\pi_0 R$. This is not true for the higher K-groups: in general, $K_{n+1}(R)$ is sensitive to the first n homotopy groups of R. The difference between the higher K-theory of ring spectra (in other words, Waldhausen K-theory) and the higher K-theory of ordinary rings (introduced by Quillen) will become important when studying the "higher" versions of the Whitehead torsion, which is our next objective.

For each $n \geq 0$, let $\mathrm{GL}_n(R)$ denote the space $\mathrm{Aut}(R^n)$ of automorphisms of R^n as an object of the ∞-category Mod_R, let $\mathrm{BGL}_n(R)$ denote its classifying space, and let $\mathrm{BGL}_\infty(R)$ be the direct limit $\varinjlim \mathrm{BGL}_n(R)$. The proof of Corollary 4 shows that there is a canonical map

$$\mathrm{BGL}_\infty(R) \to K(R)^\circ,$$

and that this map is an isomorphism on the first homology group. In fact, something stronger is true: we can identify $K(R)^\circ$ with the space obtained from $\mathrm{BGL}_\infty(R)$ by performing the "plus construction" with respect to the commutator subgroup of $\pi_1 \mathrm{BGL}_\infty(R) = \mathrm{GL}_\infty(\pi_0 R)$. To prove this, it suffices to establish the following:

Proposition 6. *Let M be an abelian group with an action of $\pi_1 K(R)^\circ$, which we view as a local system on both $K(R)^\circ$ and on $\mathrm{BGL}_\infty(R)$. Then the canonical map*

$$\mathrm{H}_*(\mathrm{BGL}_\infty(R); M) \to \mathrm{H}_*(K(R)^\circ; M)$$

is an isomorphism.

Remark 7. Proposition 6 shows that, in the case of a discrete ring R, Waldhausen K-theory (defined via the procedure we have been discussing) agrees with Quillen K-theory (defined via the plus construction).

Proof of Proposition 6. The element $1 = [R] \in K_0(R)$ determines a map of infinite loop spaces $QS^0 \to K(R)$, which induces a map

$$\mathbf{Z}/2\mathbf{Z} \simeq \pi_1 QS^0 \to \pi_1 K(R) = K_1(R).$$

Let $\epsilon \in K_1(R)$ denote the image under this map of the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ (under the isomorphism $K_1(R) \simeq \mathrm{GL}_\infty(\pi_0 R)^{\mathrm{ab}}$ of Corollary 4, this corresponds to the class represented by an odd permutation of coordinates; equivalently, it is the class represented by $-1 \in \mathrm{GL}_1(R)$, by an exercise from Lecture 3). Then ϵ is an element of $K_1(R)$ having order 2, and therefore induces an involution of the abelian group M.

Let us say that an abelian group M with an action of $K(R)^\circ$ is *good* if the conclusion of Proposition 6 holds for M. It follows immediately that if we are given an exact sequence of representations

$$0 \to M' \to M \to M'' \to 0$$

where two of the terms are good, then so is the third. We may therefore assume without loss of generality that the involution of M determined by ϵ is either id_M or $-\mathrm{id}_M$.

Let us first treat the case where the action of ϵ on M is by the identity, since this is a bit easier. Set $G = K_1(R)/\epsilon$, and regard M as a module over the group ring $\mathbf{Z}[G]$. Then we can regard the direct sum $\mathbf{Z}[G] \oplus M$ as a square-zero extension of $\mathbf{Z}[G]$. We have an exact sequence of multiplicative groups

$$0 \to M \to (\mathbf{Z}[G] \oplus M)^\times \to \mathbf{Z}[G]^\times \to 0.$$

Consequently, to prove that M is good, it will suffice to show that $(\mathbf{Z}[G] \oplus M)^\times$ and $\mathbf{Z}[G]^\times$ are good. Both of these are special cases of the following assertion:

($*$) Let A be a commutative ring and suppose we are given a map $\psi : K_1(R) \to A^\times$ which annihilates ϵ. Then A is good (with respect to the action of $K_1(R)$ via multiplication).

Let $K'(R)$ denote the fiber product $\mathbf{Z} \times_{K_0(R)} K(R)$, so that $K'(R)$ is an infinite loop space with $\pi_0 K'(R) = \mathbf{Z}$ whose connected components are homotopy equivalent to $K(R)^\circ$. Note that ψ can be identified with a map of infinite loop spaces $K(R)^\circ \to K(A^\times, 1)$, and the condition that ψ annihilates ϵ is equivalent to the statement that this map extends to an infinite loop map

$$K'(R) \to K(A^\times, 1).$$

We can regard this infinite loop map as giving a local system $\mathcal{L}$ of A-modules on $K'(R)$ which is *multiplicative* with respect to the E_∞-structure on $K'(R)$, so that $C_*(K'(R); \mathcal{L})$ can be identified with an E_∞-algebra over $\mathbf{Z}$. Note that we have a canonical map of E_∞-spaces

$$\theta : \amalg_{n \geq 0} \mathrm{BGL}_n(R) \to K'(R)$$

which induces a map of E_∞-algebras

$$C_*(\amalg_{n \geq 0} \mathrm{BGL}_n R; \theta^* \mathcal{L}) \to C_*(K'(R); \mathcal{L}).$$

Arguing as in the proof of Corollary 4, we see that this map exhibits the right hand side as obtained from the left hand side by inverting a single element $[X] \in \mathrm{H}_0(\mathrm{BGL}_1(R); \theta^* \mathcal{L})$, from which it follows that $\mathrm{H}_*(K(R)^\circ; A)$ is given by the direct limit of the sequence

$$\mathrm{H}_*(\mathrm{BGL}_0(R); A) \to \mathrm{H}_*(\mathrm{BGL}_1(R); A) \to \cdots$$

This completes the proof in the case where ϵ acts by the identity on M.

In the case where ϵ acts by $-\mathrm{id}_M$, we can use essentially the same strategy, but we need a few modifications. First, we should replace the group algebra $\mathbf{Z}[G]$ by the "twisted group algebra" obtained from $\mathbf{Z}[K_1(R)]$ by setting ϵ equal to -1. We then reduce to proving the following analogue of ($*$):

($*'$) Let A be a commutative ring and suppose we are given a map $\psi : K_1(R) \to A^\times$ which carries ϵ to -1. Then A is good (with respect to the action of $K_1(R)$ via multiplication).

Let Y be the classifying space for invertible *graded* A-modules, so that $Y \simeq \mathbf{Z} \times K(A^\times, 1)$. We regard Y as an infinite loop space using the usual commutativity constraint for tensor products of graded modules. Then multiplication by $\eta \in \pi_1 QS^0$ carries the generator $1 \in \mathbf{Z}$ to $-1 \in A^\times$. It follows that the infinite loop map $K(R)^\circ \times K(A^\times, 1)$ determined by ψ fits into a commutative diagram of E_∞-maps

$$\begin{array}{ccccc} K(R)^\circ & \longrightarrow & K'(R) & \longrightarrow & \mathbf{Z} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \downarrow \\ K(A^\times, 1) & \longrightarrow & Y & \longrightarrow & \mathbf{Z}. \end{array}$$

As before, we can identify the map $\overline{\psi}$ with a mutiplicative local system $\mathcal{L}$ on $K'(R)$: the only difference is that $\mathcal{L}$ is now a *graded* local system (on the connected component $\{n\} \times K(R)^\circ$, we regard $\mathcal{L}$ as lying in degree n). We again obtain a map of E_∞-algebras

$$C_*(\amalg_{n \geq 0} \mathrm{BGL}_n R; \theta^* \mathcal{L}) \to C_*(K'(R); \mathcal{L}).$$

which exhibits the right hand side as a localization of the left hand side, this time by inverting a class $[X] \in \mathrm{H}_1(\mathrm{BGL}_1(R); \theta^* \mathcal{L})$. Identifying the restriction of $\mathcal{L}$ to each component with a shift of the local system A, we again obtain the desired isomorphism

$$\mathrm{H}_*(K(R)^\circ; A) \simeq \varinjlim \mathrm{H}_*(\mathrm{BGL}_n(R); A).$$

□