# The Wall Finiteness Obstruction (Lecture 2)

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We begin with the following:

**Question 1.** Let $M$ be a compact manifold. Does $M$ have the homotopy type of a finite CW complex?

Of course, if $M$ is triangulable, then it is actually *homeomorphic* to a finite simplicial complex. This provides an affirmative answer 1 when $M$ is smooth (since any smooth manifold can be triangulated).

For general topological manifolds, Question 1 is not so easy to answer. As a starting point, we note that any (paracompact) topological manifold $M$ has the homotopy type of a (possibly infinite) CW complex $X$: this follows from the theory of absolute neighborhood retracts, which we will review in a subsequent lecture. Of course, we should not expect that $X$ can be chosen finite in the case where $M$ is noncompact (for a counterexample, consider the "surface of infinite genus").

Let us fix a homotopy equivalence $f : M \to X$, where $X$ is a CW complex. If $M$ is compact, then $f(M)$ is contained in a finite subcomplex $X_0 \subseteq X$.

**Exercise 2.** Prove this.

Let $g$ be a homotopy inverse to $f$. Then the composite map

$$X \xrightarrow{fg} X_0 \hookrightarrow X$$

is homotopic to the identity map from $M$ to itself. This motivates the following:

**Definition 3.** Let $X$ be a CW complex (or, more generally, a space with the homotopy type of a CW complex). We say that $X$ is *finitely dominated* if it is a retract (in the homotopy category of CW complexes) of a finite CW complex $Y$. In other words, $X$ is finitely dominated if there exists a finite CW complex $Y$ and a pair of maps

$$i : X \to Y \qquad r : Y \to X$$

such that $r \circ i$ is homotopic to the identity map from $X$ to itself.

Of course, every finite CW complex $X$ is finitely dominated: we can take $Y = X$ and the maps $i$ and $r$ to be the identity. More generally, if $X$ is homotopy equivalent to a finite CW complex, then $X$ is finitely dominated.

**Question 4.** Let $X$ be a finitely dominated CW complex. Does $X$ have the homotopy type of a finite CW complex?

We will see that the answer to Question 4 is "no" in general, but "yes" in many cases (for example, when $X$ is simply connected). Moreover, the *topological* question of deciding whether or not $X$ is finitely dominated will be reduced to an *algebraic* one (the vanishing of a certain $K$-theory class).

**Remark 5.** It is not true in general that every finitely dominated space $X$ has the homotopy type of a finite CW complex. Nevertheless, it can be shown that every compact manifold has this property: that is, Question 1 has an affirmative answer, though we will not establish that in this lecture (a proof is given in Kirby-Siebenmann; if time permits, we'll discuss a stronger result later in this course).

We begin by summarizing some of the finiteness properties enjoyed by finitely dominated spaces.

**Lemma 6.** *Let $X$ be a finitely dominated space. Then:*

($a$) *The set $\pi_0 X$ is finite.*

($b$) *For each base point $x \in X$, the group $\pi_1(X, x)$ is finitely presented.*

($c$) *Let $x \in X$ and let $X^\circ \subseteq X$ be the path component of $x$. For each abelian group $V$ with an action of the fundamental group $\pi_1(X, x)$, let $\mathrm{H}^*(X^\circ; V)$ denote the cohomology of $X^\circ$ with coefficients in the local system determined by $V$. Then the construction $V \mapsto \mathrm{H}^*(X^\circ; V)$ commutes with filtered direct limits.*

($d$) *For each $x \in X$ as above, there exists an integer $n$ such that $\mathrm{H}^*(X^\circ; V) \simeq 0$ for $* > n$ and any representation $V$ of $\pi_1(X, x)$.*

*Proof.* Choose finite CW complex $Y$ and a map $i : X \to Y$ which admits a left homotopy inverse $r$. Note that $\pi_0 Y$ is finite and that the map $\pi_0 X \to \pi_0 Y$ is injective (it has a left inverse given by $r$). This proves ($a$). To prove ($b$), we note that for each $x \in X$ the induced map $i_* : \pi_1(X, x) \to \pi_1(Y, y)$ is split injective. Since $Y$ is a finite complex, the group $\pi_1(Y, y)$ is finitely presented. It follows that the group $\pi_1(X, x)$ is finitely presented, which proves ($b$). Let $Y^\circ$ be the path component of $y$. By composition with a left inverse to $i_*$, we see that representations $V$ of $\pi_1(X, x)$ can be extended functorially to representations of $\pi_1(Y, y)$, which we can regard as local systems on $Y^\circ$. Using cellular cochains to compute $\mathrm{H}^*(Y^\circ; V)$, we see immediately that the construction $V \mapsto \mathrm{H}^*(Y^\circ; V)$ commutes with filtered direct limits. We can functorially identify $\mathrm{H}^*(X^\circ; V)$ with a direct summand of $\mathrm{H}^*(Y^\circ; V)$, so the construction $V \mapsto \mathrm{H}^*(X^\circ; V)$ also commutes with filtered direct limits. This proves ($c$), and assertion ($d$) follows if we take $n = \dim(Y)$. □

**Remark 7.** Lemma 6 actually *characterizes* finitely dominated spaces; we will give a proof at the end of this lecture.

We next show that conditions ($a$) through ($c$) guarantee that $X$ behaves approximately like a finite-dimensional CW complex.

**Proposition 8.** *Let $X$ be a CW complex which satisfies conditions ($a$), ($b$), and ($c$) of Lemma 6. For each integer $n \geq 0$, there exists a finite CW complex $Z$ of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$.*

We will deduce Proposition 8 from the following more precise statement:

**Lemma 9.** *Let $X$ be a CW complex which satisfies conditions ($a$), ($b$), and ($c$) of Lemma 6. Suppose we are given an $(n-1)$-connected map $f : Z \to X$, where $Z$ is a finite CW complex. Then there exists another finite CW complex $Z'$, obtained from $Z$ by attaching finitely many $n$-cells, and an $n$-connected map $f' : Z' \to X$ extending $f$. In particular, we have $\dim(Z') \leq \max\{n, \dim(Z)\}$.*

For the remainder of this lecture, it will be convenient to always assume that the space $X$ is connected (we can always handle disconnected spaces by considering each connected component separately). Fix a base point $x \in X$, let $G = \pi_1(X, x)$, and let $\widetilde{X}$ be a universal cover of $X$ (so that $G$ acts on $\widetilde{X}$ by deck transformations).

*Proof of Lemma 9.* If $n = 0$, then we can either take $Z' = Z$ (if $Z$ is nonempty) or $Z' = *$ (if $Z$ is empty).

We next consider the case $n = 1$. If $Z$ is not connected, then we first enlarge $Z$ by adding 1-cells connecting the different components of $Z$ (the map $f$ extends continuously over this enlargement by virtue of our assumption that $X$ is connected). Without loss of generality, we may assume that there exists a 0-cell $z \in Z$ such that $f(z) = x$. We let $Z'$ be obtained from $Z$ by attaching several loops based at the point $z$, and we define $f$ so that it carries these loops to generators of the group $G = \pi_1(X, x)$. Since $G$ is finitely generated, this only requires finitely many 1-cells.

We now consider the case $n = 2$. Let $z \in Z$ be as above, and consider the group homomorphism $\phi : \pi_1(Z, z) \to G$ induced by $f$. Since $\pi_1(Z, z)$ is finitely generated and $G$ is finitely presented, the kernel $\ker(\phi)$ is generated as a normal subgroup by finitely many elements of $\pi_1(Z, z)$. Each of these elements can be represented by a loop in the 1-skeleton of $Z$. We first enlarge $Z$ by attaching 2-cells along each of these loops (since the corresponding elements of $\pi_1(Z, z)$ are annihilated by $\phi$, the function $f$ will extend continuously). We may therefore reduce to the case where $\phi$ is an isomorphism.

We can now treat all of the cases $n \geq 2$ in a uniform manner. For any abelian group $V$ with an action of $G$, we let $\mathrm{H}_*(X, Z; V)$ and $\mathrm{H}^*(X, Z; V)$ denote the homology and cohomology of $X$ relative to $Z$ with coefficients in $V$. Since $f$ is $(n-1)$-connected, the groups $\mathrm{H}_*(X, Z; \mathbf{Z}[G])$ vanish for $* < n$. It follows from the universal coefficient theorem that we have canonical isomorphisms

$$\mathrm{H}^n(X, Z; V) \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z; \mathbf{Z}[G]), V).$$

Since $X$ satisfies condition $(c)$ of Lemma 6 and $Z$ is finite CW complex, the construction $V \mapsto \mathrm{H}^n(X, Z; V)$ commutes with filtered direct limits (exercise!)

It follows that the construction

$$V \mapsto \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z_n; \mathbf{Z}[G]), V)$$

also commutes with filtered direct limits. In particular, $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is finitely generated as a module over $\mathbf{Z}[G]$.

Set $\widetilde{Z} = Z \times_X \widetilde{X}$. Applying the relative Hurewicz theorem to the map $\widetilde{Z} \to \widetilde{X}$, we deduce that the Hurewicz map

$$\pi_n(X, Z) \simeq \pi_n(\widetilde{X}, \widetilde{Z}) \to \mathrm{H}_n(\widetilde{X}, \widetilde{Z}; \mathbf{Z}) \simeq \mathrm{H}_n(X, Z; \mathbf{Z}[G])$$

is an isomorphism. Consequently, the group $\pi_n(X, Z)$ is finitely generated as a $\mathbf{Z}[G]$-module. Each element of $\pi_n(X, Z)$ supplies a recipe for attaching an $n$-cell to $Z$ and extending the definition of $f$ over that $n$-cell. Without loss of generality, we may assume that the relevant attaching maps factor through the $(n-1)$-skeleton of $Z$. Let $Z'$ be the CW complex obtained from $Z$ by attaching $n$-cells corresponding to a set of generators of $\pi_n(X, Z)$, so that $f$ extends to an $n$-connected map $f' : Z' \to X$. □

For any space $X$ satisfying conditions $(a)$, $(b)$, and $(c)$ of Lemma 6, Lemma 9 allows us to construct a sequence of better and better approximations to $X$. It is condition $(d)$ that will allow us to stop this construction.

**Definition 10.** Let $X$ be a CW complex and let $n \geq 2$ be an integer. We will say that $X$ has *homotopy dimension* $\leq n$ if it satisfies condition $(d)$ of Lemma 6: that is, if $\mathrm{H}^*(X; \mathcal{L})$ vanishes for $* > n$ and any local system of abelian groups $\mathcal{L}$ on $X$.

**Remark 11.** Definition 10 makes sense for any value of $n$, but is not really the right condition when $n = 0$ and $n = 1$: in those cases, one should also require vanishing for "nonabelian" cohomology.

**Lemma 12.** *Let $X$ be a CW complex satisfying the conditions of Lemma 6. Let $Z$ be a finite CW complex of dimension $\leq n-1$ and let $f : Z \to X$ be an $(n-1)$-connected map. If $X$ has homotopy dimension $\leq n$, then the homology group $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is a finitely generated projective $\mathbf{Z}[G]$-module.*

*Proof.* Let $V$ be any abelian group with an action of $G$, which determines local systems on $X$ and $Z$ which we will also denote by $V$. Since $Z$ is $(n-1)$-dimensional, the local cohomology groups $\mathrm{H}^*(Z; V)$ vanish for $* \geq n$. Using the exact sequence

$$\mathrm{H}^{*-1}(Z; V) \to \mathrm{H}^*(X, Z; V) \to \mathrm{H}^*(X; V),$$

we see that the groups $\mathrm{H}^*(X, Z; V)$ vanish for $* > n$. Any exact sequence of representations $0 \to V' \to V \to V'' \to 0$ gives rise to a long exact sequence

$$\mathrm{H}^n(X, Z; V') \to \mathrm{H}^n(X, Z; V) \to \mathrm{H}^n(X, Z; V'') \to \mathrm{H}^{n+1}(X, Z; V') \simeq 0$$

It follows that the construction

$$V \mapsto \mathrm{H}^n(X, Z; V) \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_n(X, Z; \mathbf{Z}[G]); V)$$

is a *right exact* functor of $V$, so that $\mathrm{H}_n(X, Z_{n-1}; \mathbf{Z}[G])$ is a projective $\mathbf{Z}[G]$-module. As in the proof of Lemma 9, it is finitely generated because the construction $V \mapsto \mathrm{H}^n(X, Z; V)$ commutes with filtered direct limits. □

**Remark 13.** In the situation of Lemma 12, the relative homology groups $\mathrm{H}_*(X, Z; \mathbf{Z}[G])$ vanish for $* \neq n$. For $* < n$, this follows from our connectivity assumption on the map $f : Z \to X$. On the other hand, suppose there were some $m > n$ for which $\mathrm{H}_m(X, Z; \mathbf{Z}[G]) \neq 0$. Choose $m$ as small as possible and set $A = \mathrm{H}_m(X, Z; \mathbf{Z}[G])$. Using the projectivity of $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$, the universal coefficient formula gives

$$\mathrm{H}^m(X, Z; A) = \mathrm{Hom}_{\mathbf{Z}[G]}(\mathrm{H}_m(X, Z); \mathbf{Z}[G]), A) = \mathrm{Hom}_{\mathbf{Z}[G]}(A, A) \neq 0.$$

This is impossible, since $\mathrm{H}^m(X; A)$ and $\mathrm{H}^{m-1}(Z; A)$ both vanish.

Of course, the projective module $P = \mathrm{H}_n(X, Z; \mathbf{Z}[G])$ depends on the choice of $(n-1)$-connected map $f : Z \to X$. For example, we could enlarge the CW complex $Z$ by adjoining some several $(n-1)$-spheres which map trivially to $X$; this would have the effect of replacing $P$ by a direct sum $P \oplus \mathbf{Z}[G]^r$, where $r$ is the number of additional spheres. This motivates the following:

**Definition 14.** Let $R$ be a ring and let $P$ be a finitely generated $R$-module. We say that $P$ is *stably free* if $P \oplus R^a$ is a free $R$-module for some $a \geq 0$.

**Proposition 15.** *Let $n \geq 3$ and let $X$ be a space which satisfies the conditions of Lemma 6 which is of homotopy dimension $\leq n$. Choose a finite CW complex $Z$ of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$. Then the following conditions are equivalent:*

- $(i)$ *The space $X$ is homotopy equivalent to a finite CW complex of dimension $\leq n$.*
- $(ii)$ *The projective module $P = \mathrm{H}_n(X, Z; \mathbf{Z}[G])$ is stably free.*

*Proof.* Suppose first that $P$ is stably free. As indicated above, we can then alter the definition of $Z$ (attaching some extra spheres) to arrange that $P$ is actually free. In this case, we repeat the construction of Lemma 9 but slightly more carefully: we choose a basis for $\mathrm{H}_n(X, Z; \mathbf{Z}[G]) \simeq \pi_n(X, Z)$ and attach only $n$-cells corresponding to those basis elements. This produces a map $f' : Z' \to X$ which is an isomorphism on fundamental groups where the relative homology $\mathrm{H}_*(X, Z; \mathbf{Z}[G]) \simeq \mathrm{H}_*(\widetilde{X}, \widetilde{Z}'; \mathbf{Z})$ vanishes, so that $f'$ is a homotopy equivalence by Whitehead's theorem.

Conversely, suppose that $X$ is homotopy equivalent to a finite CW complex of dimension $n$. Without loss of generality we may assume that the map $f$ is cellular, so that the relative homology $\mathrm{H}_n(X, Z; \mathbf{Z}[G])$ can be computed by a cellular chain complex of finitely generated free $\mathbf{Z}[G]$-modules

$$0 \to \mathbf{Z}[G]^{r_n} \to \mathbf{Z}[G]^{r_{n-1}} \to \cdots \to \mathbf{Z}[G]^{r_0} \to 0.$$

Since this complex is acyclic away from the top degree, it is split exact: that is, it has the form

$$0 \to Q_n \oplus Q_{n-1} \to Q_{n-1} \oplus Q_{n-2} \to \cdots \to Q_1 \oplus Q_0 \to Q_0 \to 0.$$

It follows by induction on $i$ that each $Q_i$ is stably free; in particular $P = Q_n$ is stably free. □

**Remark 16.** I believe it is an open question whether Proposition 15 is also valid for $n = 2$ (the proof given above certainly does not apply).

**Corollary 17.** *Let $X$ be a finitely dominated space which is simply connected. Then $X$ has the homotopy type of a finite CW complex.*

*Proof.* Every finitely generated projective $\mathbf{Z}$-module is free. $\square$

Proposition 15 allows us to quantify the failure of finitely dominated spaces to be homotopy equivalent to finite CW complexes.

**Definition 18.** Let $R$ be a ring. We let $K_0(R)$ denote the Grothendieck group of projective $R$-modules: that is, the free abelian group generated by symbols $[P]$, where $P$ is a projective $R$-module, modulo the relations

$$[P] = [P'] + [P'']$$

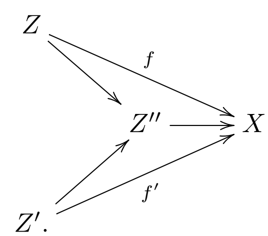
when there exists an isomorphism $P \simeq P' \oplus P''$.

The construction $n \mapsto n[R]$ determines a group homomorphism $\mathbf{Z} \to K_0(R)$. We let $\widetilde{K}_0(R)$ denote the cokernel of this homomorphism. We refer to $\widetilde{K}_0(R)$ as the *reduced $K$-group of $R$*.

**Proposition 19.** *Let $X$ be a finitely dominated space of homotopy dimension $\leq n$. Let $Z$ be a finite CW complex of dimension $< n$ and let $f : Z \to X$ be an $(n-1)$-connected map. Then the image of the class $[\mathrm{H}_n(X, Z; \mathbf{Z}[G])]$ in the reduced $K$-group $\widetilde{K}_0(\mathbf{Z}[G])$ does not depend on the choice of $Z$ or $f$.*

*Proof.* Let $Z'$ be another finite CW complex of dimension $< n$ equipped with an $(n-1)$-connected map $f' : Z' \to X$. We wish to show that $[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])]$ in the group $\widetilde{K}_0(\mathbf{Z}[G])$.

Starting with the map $Z \amalg Z' \to X$ and repeatedly applying Lemma 9, we obtain a homotopy commutative diagram




It will therefore suffice to show that we have equalities

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z''; \mathbf{Z}[G])] \qquad [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])].$$

In other words, we may replace $Z'$ by $Z''$ and thereby reduce to the case where the map $f$ factors as a composition

$$Z \xrightarrow{g} Z' \xrightarrow{f'} X.$$

Note that the map $g$ is automatically $(n-2)$-connected. Using Lemma 9, we see that $g$ factors as a composition

$$Z \xrightarrow{g'} Z^+ \xrightarrow{g''} Z'$$

where $g''$ is $(n-1)$-connected and $Z^+$ is obtained from $Z$ by attaching finitely many $(n-1)$-cells. Replacing $g$ by $g'$ or $g''$, we are reduced to two special cases:

($a$) The map $g$ is $(n-1)$-connected. In this case, we have a short exact sequence

$$0 \to \mathrm{H}_n(Z', Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z'; \mathbf{Z}[G]) \to 0$$

(the exactness on the left follows from Remark 13). This sequence splits (since $\mathrm{H}_n(X, Z'; \mathbf{Z}[G])$ is projective), so we have

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] = [\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] + [\mathrm{H}_n(Z', Z; \mathbf{Z}[G])].$$

It will therefore suffice to show that the class $[\mathrm{H}_n(Z', Z; \mathbf{Z}[G])]$ vanishes in $\widetilde{K}_0(\mathbf{Z}[G])$. This follows from Proposition 15, since $Z$ is a finite CW complex.

(*b*) The CW complex $Z'$ is obtained from $Z$ by attaching $(n-1)$-cells. In this case, we have an exact sequence

$$0 \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z'; \mathbf{Z}[G]) \to \mathrm{H}_{n-1}(Z', Z; \mathbf{Z}[G]) \to 0$$

which gives

$$\begin{aligned}
[\mathrm{H}_n(X, Z'; \mathbf{Z}[G])] &= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathrm{H}_{n-1}(Z', Z; \mathbf{Z}[G])] \\
&= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathbf{Z}[G]^r] \\
&= [\mathrm{H}_n(X, Z; \mathbf{Z}[G])].
\end{aligned}$$

□

Proposition 19 motivates the following:

**Definition 20.** Let $X$ be a finitely dominated space. Choose an integer $n \geq 2$ such that $X$ has homotopy dimension $\leq n$, a CW complex $Z$ of dimension $< n$, and an $(n-1)$-connected map $f : Z \to X$. The *Wall finiteness obstruction* of $X$ is the element

$$w(X) = (-1)^n [\mathrm{H}_n(X, Z; \mathbf{Z}[G])] \in \widetilde{K}_0(\mathbf{Z}[G]).$$

**Proposition 21.** *Let $X$ be a finitely dominated space. Then the Wall finiteness obstruction $w(X)$ is well-defined.*

*Proof.* We have already seen that $w(X)$ does not depend on the map $f : Z \to X$. We now check that it is independent of $n$. Let us temporarily denote $w(X)$ by $w_n(X)$ to emphasize its hypothetical dependence on $n$. Choose any integer $n \geq 2$ such that $X$ has homotopy dimension $\leq n$; we will show that $w_n(X) = w_{n+1}(X)$. To prove this, choose a CW complex $Z$ of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$. Using Lemma 9, we can extend $f$ to an $n$-connected map $f' : Z' \to X$ where $Z'$ is obtained from $Z$ by attaching finitely many $n$-cells. We then have a (split) short exact sequence

$$0 \to \mathrm{H}_{n+1}(X, Z'; \mathbf{Z}[G]) \to \mathrm{H}_n(Z, Z'; \mathbf{Z}[G]) \to \mathrm{H}_n(X, Z; \mathbf{Z}[G]) \to 0$$

which gives the relation

$$[\mathrm{H}_n(X, Z; \mathbf{Z}[G])] + [\mathrm{H}_{n+1}(X, Z'; \mathbf{Z}[G])] = [\mathrm{H}_n(Z, Z'; \mathbf{Z}[G])] = 0 \in \widetilde{K}_0(\mathbf{Z}[G]).$$

□

By virtue of Proposition 15, the finiteness obstruction $w(X)$ is zero if and only if $X$ has the homotopy type of a finite CW complex.

We conclude this lecture by tying up a few loose ends. We start with a converse to Lemma 6.

**Proposition 22.** *Let $X$ be a CW complex satisfying conditions (a) through (d) of Lemma 6. Then $X$ is finitely dominated.*

*Proof Sketch.* Write $X$ as a union of finite subcomplexes $X_\alpha$. It will suffice to show that the identity map $\mathrm{id} : X \to X$ is homotopic to a map which factors through some $X_\alpha$. We can replace each of the inclusions $X_\alpha \to X$ by a fibration $p_\alpha : E_\alpha \to X$; we wish to show that one of these inclusions has a section.

For each integer $m$, let $\tau_{\leq m} E_\alpha$ denote the $m$th stage in the relative Postnikov tower of $E_\alpha$ over $X$ (so that we have a fibration $p_{\alpha,m} : \tau_{\leq m} E_\alpha \to X$ whose fibers have no homotopy groups above $m$). Suppose we are given a section $s_m$ of some $p_{\alpha,m}$. Note that if $m \geq 1$, then the fiber product

$$(\tau_{\leq m+1} E_\alpha) \times_{E_\alpha} X$$

is a fibration over whose fibers have the form $K(A_{x,\alpha}, m+1)$, where $x \mapsto A_{x,\alpha}$ is a local system of abelian groups on $X$. Consequently, the obstruction to lifting $s_m$ to a section of $p_{\alpha,m+1}$ is measured by a cohomology class $\eta(s_m) \in \mathrm{H}^{m+2}(X; A_\alpha)$. If $m+2$ is larger than the homotopy dimension of $X$ (which is finite by assumption), then $\eta(s_m)$ automatically vanishes, so any section of $p_{\alpha,m}$ can be lifted to a section of $p_\alpha$.

We will complete the proof by showing that for every integer $m$, there exists an index $\alpha$ such that $p_{\alpha,m}$ admits a section. The proof proceeds by induction on $m$. Suppose first that $m \geq 1$ and that we are given a section $s_m$ as above. We claim that it is possible to choose $\beta \geq \alpha$ such that the image of $\eta(s_m)$ vanishes in $\mathrm{H}^{m+2}(X; A_\beta)$. In fact, we claim that the direct limit $\varinjlim_{\beta \geq \alpha} \mathrm{H}^{m+2}(X; A_\beta)$ vanishes. Since $X$ satisfies condition $(c)$ of Lemma 6, it will suffice to show that the direct limit $\varinjlim_{\beta \geq \alpha} A_\beta$ vanishes as a local system of abelian groups on $X$. This follows immediately from the fact that $X$ is a homotopy colimit of the diagram $\{E_\alpha\}$.

It remains to treat small values of $m$. Let us begin with the case $m = 0$, so that each $\tau_{\leq m} E_\alpha$ can be regarded as a covering space of $X$. Let $S_\alpha$ denote the fiber over the base point $x \in X$, so that each $S_\alpha$ is a set with an action of the group $G$. To choose a section of $\tau_{\leq m} E_\alpha$, we must show that $S_\alpha$ contains an element which is fixed by $G$. Because $G$ is finitely generated, passage to $G$-invariants commutes with filtered direct limits. It will therefore suffice to show that the direct limit $\varinjlim S_\alpha$ contains an element which is fixed by $G$. This is clear, since $\varinjlim S_\alpha$ consists of a single point (it is $\pi_0$ of the homotopy fiber of the identity map $X \to X$)

We conclude by treating the case $m = 1$. Let us assume that there exists an index $\alpha$ and that we have chosen a section of the map $\tau_{\leq 0} E_\alpha \to X$. For each $\beta \geq \alpha$, let $E'_\beta$ denote the fiber product $\tau_{\leq 1} E_\beta \times_{\tau_{\leq 0} E_\beta} X$. The projection map $q_\beta : E'_\beta \to X$ is a fibration whose fibers have the form $K(\Pi; 1)$. Let $G_\beta$ denote the fundamental group of $E'_\beta$, so that we have $\varinjlim G_\beta \simeq G$. Since $G$ is finitely presented, it follows that the natural map $G_\beta \to G$ admits a section for $\beta$ sufficiently large. We are therefore reduced to finding a section of the induced map $E'_\beta \times_{BG_\beta} BG \to X$. This is a fibration whose fibers are of the form $K(\Pi, 1)$ where $\Pi$ is abelian, and is therefore classified by an element of $\mathrm{H}^2(X; \mathcal{L}_\beta)$ for some local system of abelian groups $\mathcal{L}_\beta$ on $X$. As before, we have $\varinjlim_\beta \mathcal{L}_\beta = 0$ so (by virtue of condition $(c)$) the direct limit $\varinjlim_\beta \mathrm{H}^2(X; \mathcal{L}_\beta)$ vanishes, and therefore the fibration is trivial for $\beta$ sufficiently large. □

**Remark 23.** Let $G$ be a finitely presented group. Then every class $\eta \in \widetilde{K}_0(\mathbf{Z}[G])$ arises as the Wall finiteness obstruction of some finitely dominated space $X$ with $\pi_1 X = G$. To see this, we first choose a connected finite 2-dimensional CW complex $X_0$ with $\pi_1 X_0 = G$. Let $\eta = [P]$ where $P$ is a finitely generated projective $\mathbf{Z}[G]$-module, so that $P$ appears as a direct summand of some free $\mathbf{Z}[G]$-module $\mathbf{Z}[G]^r$. Then $P$ is the image of an idempotent map $e : \mathbf{Z}[G]^r \to \mathbf{Z}[G]^r$. Choose an even integer $n \geq 2$ and let $Y$ be the CW complex obtained from $X_0$ by adding $r$ $n$-cells with trivial attaching maps. Using the relative Hurewicz theorem we deduce that the relative homotopy group $\pi_n(Y, X_0)$ is isomorphic to the free module $\mathbf{Z}[G]^r$. We can therefore choose a map $\overline{e} : Y \to Y$ which is the identity on $X_0$ and which induces the idempotent endomorphism $e$ of $\mathrm{H}_n(Y, X_0; \mathbf{Z}[G]) \simeq \mathbf{Z}[G]^r$. Using the fact that $\overline{e}$ is an idempotent in the homotopy category, one can show that the homotopy colimit $X$ of the diagram

$$Y \xrightarrow{\overline{e}} Y \xrightarrow{\overline{e}} Y \xrightarrow{\overline{e}} \cdots$$

satisfies the conditions of Lemma 6 and is therefore a finitely dominated space of homotopy dimension $\leq n$; a simple calculation shows that the composite map $X_0 \hookrightarrow Y \to X$ is $(n-1)$-connected and that the relative homology $\mathrm{H}_n(X, X_0; \mathbf{Z}[G])$ is isomorphic to $P$.