Algebraic K-Theory of Ring Spectra (Lecture 19)

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Let R be an associative ring spectrum (here by associative we mean A_∞ or associative up to *coherent* homotopy; homotopy associativity is not sufficient for the considerations which follow). Then we can consider the ∞-category Mod_R whose objects are (right) R-module spectra. We say that an R-module M is *perfect* if it is a compact object of Mod_R. We let $\mathrm{Mod}_R^{\mathrm{perf}}$ denote the full subcategory of Mod_R spanned by the perfect R-modules.

Remark 1. The full subcategory $\mathrm{Mod}_R^{\mathrm{perf}}$ can be characterized as the smallest full subcategory of Mod_R which contains R and is closed under finite colimits, desuspensions, and passage to direct summands.

Example 2. Every associative ring R can be regarded as an associative ring spectrum (by identifying R with its Eilenberg-MacLane spectrum HR). In this case, the ∞-category Mod_R has a concrete algebraic description: its objects can be identified with chain complexes of (ordinary) R-modules, and the objects of $\mathrm{Mod}_R^{\mathrm{perf}}$ can be identified with bounded chain complexes of finitely generated projective R-modules.

The ∞-categories Mod_R and $\mathrm{Mod}_R^{\mathrm{perf}}$ are stable. In particular, $\mathrm{Mod}_R^{\mathrm{perf}}$ is a pointed ∞-category which admits finite colimits, so we can consider its K-theory.

Definition 3. Let R be an associative ring spectrum. We set $K(R) = K(\mathrm{Mod}_R^{\mathrm{perf}})$. We will refer to $K(R)$ as the *algebraic K-theory space* of R.

Remark 4. It may appear that Definition 3 is a very special case of the construction described in Lecture 16. However, this is not really the case: the K-theory of an arbitrary pointed ∞-category $\mathcal{C}$ which admits finite colimits can be described in terms of the K-theory of ring spectra. This can be seen as follows:

- (a) Since the K-theory of $\mathcal{C}$ is the same as the K-theory of its Spanier-Whitehead ∞-category $SW(\mathcal{C})$, we might as well assume that $\mathcal{C}$ is stable.
- (b) In the previous lecture, we showed that replacing $\mathcal{C}$ by its idempotent completion has little effect on the space $K(\mathcal{C})$ (it only changes the set of connected components). We therefore might as well assume that $\mathcal{C}$ is idempotent complete.
- (c) We will say that an object $C \in \mathcal{C}$ is a *generator* if the smallest stable subcategory of $\mathcal{C}$ which contains C and is idempotent complete is $\mathcal{C}$ itself. In general, there need not exist a generator for $\mathcal{C}$: however, we can always write $\mathcal{C}$ as a filtered union $\bigcup \mathcal{C}_\alpha$, where each $\mathcal{C}_\alpha$ is a stable subcategory which has a generator. Then $K(\mathcal{C}) \simeq \varinjlim K(\mathcal{C}_\alpha)$. We therefore might as well assume that $\mathcal{C}$ has a generator.
- (d) For any object $C \in \mathcal{C}$, the sequence of spaces $\{\mathrm{Map}_{\mathcal{C}}(C, \Sigma^n C)\}_{n \geq 0}$ determine a spectrum which we will denote by $\mathrm{End}(C)$. One can show that $\mathrm{End}(C)$ is a ring spectrum, and that there is a fully faithful embedding
$$\mathrm{Mod}_{\mathrm{End}(C)}^{\mathrm{perf}} \hookrightarrow \mathcal{C}$$
$$M \mapsto M \wedge_{\mathrm{End}(C)} C$$
which is an equivalence if and only if C is a generator. If this condition is satisfied, we then have $K(\mathcal{C}) \simeq K(\mathrm{End}(C))$.

Let R be an associative ring spectrum and let M be an R-module. We will say that M is *finitely generated and projective* if it can be realized as a direct summand of R^n for some n. We let $\mathrm{Mod}_R^{\mathrm{proj}}$ denote the full subcategory of Mod_R spanned by the finitely generated projective R-modules. Note that this subcategory is contained in $\mathrm{Mod}_R^{\mathrm{perf}}$. The ∞-category $\mathrm{Mod}_R^{\mathrm{proj}}$ has finite coproducts (though it does not have finite colimits in general), so we can define its additive K-theory $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$ as in the previous lecture (here we will think of $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}})$ as a space, rather than a spectrum). Our goal in this lecture is to prove the following result:

Theorem 5. *Let R be a* connective *ring spectrum (meaning that $\pi_n R \simeq 0$ for $n < 0$). Then the inclusion $\mathrm{Mod}_R^{\mathrm{proj}} \hookrightarrow \mathrm{Mod}_R^{\mathrm{perf}}$ induces a homotopy equivalence of K-theory spaces*

$$K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{proj}}) \to K(\mathrm{Mod}_R^{\mathrm{perf}}) = K(R).$$

Theorem 5 will allow us to get a concrete handle on $K(R)$ (and describe some of its homotopy groups) in the case where R is connective; we will return to this point in the next lecture.

To prove Theorem 5, we will need to introduce a family of intermediate objects which interpolate between $\mathrm{Mod}_R^{\mathrm{proj}}$ and $\mathrm{Mod}_R^{\mathrm{perf}}$.

Definition 6. Let M be an R-module and let n be an integer. We say that M is *n-connective* if $\pi_m M \simeq 0$ for $m < n$. We say that M has *projective amplitude* $\leq n$ if, for every $(n+1)$-connective R-module N, every morphism $f : M \to N$ is nullhomotopic.

The hypotheses of n-connectivity and projective amplitude $\leq m$ are of a complementary nature: as m and n grow, the first condition gets stronger and the last condition gets weaker. Note that if M is n-connective and of projective amplitude $< n$, then the identity map $\mathrm{id} : M \to M$ is nullhomotopic and therefore $M \simeq 0$.

Lemma 7. *Let $M \in \mathrm{Mod}_R^{\mathrm{perf}}$. The following conditions are equivalent:*

(1) *The R-module M is finitely generated and projective.*

(2) *The R-module M is 0-connective and of projective amplitude ≤ 0.*

Proof. The implication (1) $\Rightarrow$ (2) is easy. The converse depends on a few basic facts about perfect modules. For every ordinary module N over the ring $\pi_0 R$, we can regard the Eilenberg-MacLane spectrum HN as a module over R. If M is 0-connective, then the space $\mathrm{Map}_{\mathrm{Mod}_R}(M, HN)$ is homotopy equivalent to the discrete set $\mathrm{Hom}_{\pi_0 R}(\pi_0 M, N)$ of module homomorphisms from $\pi_0 M$ into N. If M is perfect, then the construction $N \mapsto \mathrm{Hom}_{\pi_0 R}(\pi_0 M, N)$ commutes with filtered colimits and therefore $\pi_0 M$ is finitely presented as a module over $\pi_0 R$. In particular, we can choose a map $e : R^n \to M$ which is surjective on π_0. Form a cofiber sequence

$$R^n \xrightarrow{e} M \xrightarrow{f} N.$$

The assumption that e is surjective on π_0 ensures that N is 1-connective. If M has projective amplitude ≤ 0, then the map f must be nullhomotopic; a choice of nullhomotopy supplies a section of e which exhibits M as a direct summand of R^n. □

For each integer $n \geq 0$, let $\mathrm{Mod}_R^{(n)}$ denote the full subcategory of $\mathrm{Mod}_R^{\mathrm{perf}}$ spanned by those perfect R-modules M which are 0-connective and of projective amplitude $\leq n$; we let $\mathrm{Mod}_R^{(\infty)} = \bigcup_{n \geq 0} \mathrm{Mod}_R^{(n)}$. We will say that a morphism $f : M' \to M$ in $\mathrm{Mod}_R^{(n)}$ is a *cofibration* if the cofiber of f (formed in the ∞-category Mod_R) also belongs to $\mathrm{Mod}_R^{(n)}$.

Exercise 8. (*a*) Let n be an integer, and suppose we are given a cofiber sequence

$$M' \to M \to M''$$

in Mod_R. Show that if M' and M'' are n-connective (of projective amplitude $\leq n$), then M is also n-connective (of projective amplitude $\leq n$).

(*b*) Show that an R-module M is n-connective (of projective amplitude $\leq n$) if and only if the suspension $\Sigma(M)$ is $(n+1)$-connective (of projective amplitude $\leq n$).

By "rotating" cofiber sequences, we can deduce several formal consequences of (*a*) and (*b*):

(*c*) Suppose we are given a cofiber sequence

$$M' \to M \to M''.$$

Show that if M is n-connective and M' is $(n-1)$-connective, then M'' is n-connective. On the other hand, if M is n-connective and M'' is $(n+1)$-connective, then M' is n-connective. Similar statements hold if we replace "n-connective" with "of projective amplitude $\leq n$".

Exercise 9. Show that the notion of cofibration defined above endows $\mathrm{Mod}_R^{(n)}$ with the structure of an ∞-category with cofibrations in the sense of Lecture 18.

It follows from Exercise 8 that if $f : M' \to M$ is a morphism in $\mathrm{Mod}_R^{(n)}$, then $\mathrm{cofib}(f)$ is automatically connective and has projective amplitude at most $\leq n+1$.

Exercise 10. It is somewhat easier to think about cofibration *sequences* in $\mathrm{Mod}_R^{(n)}$ rather than individual cofibrations. The data of a cofibration $f : M' \to M$ in $\mathrm{Mod}_R^{(n)}$ is equivalent to the data of a cofibration sequence

$$M' \to M \to M''$$

in Mod_R where $M', M, M'' \in \mathrm{Mod}_R^{(n)}$. Note that if M and M'' belong to $\mathrm{Mod}_R^{(n)}$, then M' automatically has projective amplitude $\leq n$; it is connective if and only if the map $\pi_0 M \to \pi_0 M''$ is a surjection.

Example 11. When $n = 0$, we are considering cofibration sequences

$$M' \to M \to M''$$

where M', M, and M'' are finitely generated projective R-modules (Lemma 7). It follows that $\mathrm{Mod}_R^{(0)}$ has only split cofibrations.

Example 12. When $n = \infty$, every map in $\mathrm{Mod}_R^{(n)}$ is a cofibration.

We now turn to the proof of Theorem 5. We are interested in studying the composite map

$$K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{perf}}) \to K(\mathrm{Mod}_R^{(0)}) \to K(\mathrm{Mod}_R^{(1)}) \to \cdots \to K(\mathrm{Mod}_R^{(\infty)}) \to K(\mathrm{Mod}_R^{\mathrm{perf}}).$$

We make the following observations:

(*a*) The map $K_{\mathrm{add}}(\mathrm{Mod}_R^{\mathrm{perf}}) \to K(\mathrm{Mod}_R^{(0)})$ is a homotopy equivalence. This follows from the result we proved in Lecture 18, since every cofibration sequence in $\mathrm{Mod}_R^{(0)}$ splits.

(*b*) The ∞-category $\mathrm{Mod}_R^{\mathrm{perf}}$ can be identified with the Spanier-Whitehead ∞-category of $\mathrm{Mod}_R^{(\infty)}$ (this follows from the observation that any perfect R-module M is $(-n)$-connective for $n \gg 0$). It follows from our work in Lecture 17 that the map $K(\mathrm{Mod}_R^{(\infty)}) \to K(\mathrm{Mod}_R^{\mathrm{perf}})$ is a homotopy equivalence.

It will therefore suffice to prove the following:

Proposition 13. *For each integer $n > 0$, the canonical map $K(\mathrm{Mod}_R^{(n-1)}) \to K(\mathrm{Mod}_R^{(n)})$ is a homotopy equivalence.*

To prove Proposition 13, we will need to introduce an auxiliary construction. Let $\mathcal{C}$ be the ∞-category whose objects are cofiber sequences

$$M' \to M \to M'',$$

where $M' \in \mathrm{Mod}_R^{(n-1)}$, $M \in \mathrm{Mod}_R^{\mathrm{proj}}$, and $M'' \in \mathrm{Mod}_R^{(n)}$. We regard $\mathcal{C}$ as an ∞-category with cofibrations, where a cofibration in $\mathcal{C}$ is a map of cofiber sequences whose cofiber (formed in the ∞-category of all cofiber sequences in Mod_R) also belongs to $\mathcal{C}$.

There are evident evaluation maps

$$e' : \mathcal{C} \to \mathrm{Mod}_R^{(n-1)} \qquad e : \mathcal{C} \to \mathrm{Mod}_R^{\mathrm{proj}} \qquad e'' : \mathcal{C} \to \mathrm{Mod}_R^{(n)}.$$

These evaluation maps induce maps of K-theory spaces, which we will denote by e'_*, e_*, and e''_*. We first prove the following:

Lemma 14. *The maps e'_* and e_* induce a homotopy equivalence*

$$K(\mathcal{C}) \to K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}}).$$

Proof. We define functors $i : \mathrm{Mod}_R^{(n-1)} \to \mathcal{C}$ and $j : \mathrm{Mod}_R^{\mathrm{proj}} \to \mathcal{C}$ by the formulae

$$i(M') = (M' \to 0 \to \Sigma(M'))$$

$$j(M) = (0 \to M \to M).$$

It is clear that $e'_* i_*$ and $e''_* j_*$ are homotopic to the identity maps on $K(\mathrm{Mod}_R^{(n-1)})$ and $K(\mathrm{Mod}_R^{\mathrm{proj}})$, respectively. To complete the proof, it will suffice to show that the sum $i_* e'_* + j_* e''_*$ is homotopic to the identity map on $K(\mathcal{C})$. This follows by applying the additivity theorem to the natural cofiber sequence

$$\begin{array}{ccccc}
0 & \longrightarrow & M & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
M' & \longrightarrow & M & \longrightarrow & M'' \\
\downarrow & & \downarrow & & \downarrow \\
M' & \longrightarrow & 0 & \longrightarrow & \Sigma(M')
\end{array}$$

(actually we need a slightly more general form of the additivity theorem than the one we proved in Lecture 17, which applies to the K-theory ∞-categories with cofibrations; however, this more general statement can be proven by the same argument). □

Let $k : \mathrm{Mod}_R^{\mathrm{proj}} \to \mathcal{C}$ be the functor given by

$$k(M) = (M \to M \to 0).$$

The composite map

$$K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}}) \xrightarrow{i_*, k_*} K(\mathcal{C}) \xrightarrow{e'_*, e_*} K(\mathrm{Mod}_R^{(n-1)}) \times K(\mathrm{Mod}_R^{\mathrm{proj}})$$

is upper triangular and therefore a homotopy equivalence. It follows that i_* induces a homotopy equivalence $K(\mathrm{Mod}_R^{(n-1)}) \to \mathrm{cofib}(k_*)$, where the cofiber is formed in the ∞-category of grouplike E_∞-spaces. Since $e'' \circ k$ is nullhomotopic, we obtain a map

$$\theta : \mathrm{cofib}(k_*) \to K(\mathrm{Mod}_R^{(n)}).$$

The composite map

$$K(\mathrm{Mod}_R^{(n-1)}) \simeq \mathrm{cofib}(k_*) \xrightarrow{\theta} K(\mathrm{Mod}_R^{(n)})$$

is given concretely by the construction

$$\mathrm{Mod}_R^{(n-1)} \to \mathrm{Mod}_R^{(n)}$$
$$M' \to \Sigma(M'),$$

and therefore agrees up to a sign with the map appearing in Proposition 13. It will therefore suffice to show that θ is a homotopy equivalence.

Unwinding the definitions, we see that θ is given by a map

$$\Omega(|S_\bullet \mathcal{C}|/|S_\bullet \mathrm{Mod}_R^{\mathrm{proj}}|) \to \Omega|S_\bullet \mathrm{Mod}_R^{(n)}|,$$

where the quotient means we are forming a bar construction. Since the formation of bar constructions commutes with geometric realization, θ is obtained by looping the geometric realization of a map of simplicial spaces

$$\theta_\bullet : S_\bullet \mathcal{C} / S_\bullet \mathrm{Mod}_R^{\mathrm{proj}} \to S_\bullet \mathrm{Mod}_R^{(n)}.$$

It will therefore suffice to show that this map is a homotopy equivalence of simplicial spaces. We will take this up in the next lecture.

References