Additive K-Theory (Lecture 18)

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Let $\mathcal{C}$ be a pointed ∞-category ∞-category which admits finite colimits, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite colimits, and assume that every object of $\mathcal{C}$ can be obtained as a retract of an object of $\mathcal{C}_0$. In Lectures 14 and 15, we saw that there can be a big difference between $K_0(\mathcal{C})$ and $K_0(\mathcal{C}_0)$: in the stable case, an object $C \in \mathcal{C}$ belongs to (the essential image) of $\mathcal{C}_0$ if and only if the class $[C] \in K_0(\mathcal{C})$ belongs to the image of the map $K_0(\mathcal{C}_0) \hookrightarrow K_0(\mathcal{C})$. Our first goal in this lecture is to show that the difference between $\mathcal{C}$ and $\mathcal{C}_0$ disappears when we look at *higher* K-groups. More precisely, we have the following result:

Proposition 1. *Let $\mathcal{C}$ and $\mathcal{C}_0$ be as above. Then the canonical map $K_n(\mathcal{C}_0) \to K_n(\mathcal{C})$ is an isomorphism for $n > 0$. In other words, the diagram of spaces*

$$\begin{array}{ccc} K(\mathcal{C}_0) & \longrightarrow & K(\mathcal{C}) \\ \downarrow & & \downarrow \\ K_0(\mathcal{C}_0) & \longrightarrow & K_0(\mathcal{C}) \end{array}$$

is a homotopy pullback square.

To prove Proposition 1, we may assume without loss of generality that $\mathcal{C}$ and $\mathcal{C}_0$ are stable (since the construction $\mathcal{C} \mapsto SW(\mathcal{C})$ has no effect on K-theory). For every group G, let $B_\bullet(G)$ denote the simplicial set which models the classifying space of G, so that the set $B_n(G)$ of n-simplices of $B_\bullet(G)$ can be identified with G^n. Let us describe this in a way that makes the simplicial structure of $B_\bullet G$ more apparent. Using additive notation for the group structure on G (we will ultimately be interested in the case where G is abelian), we can identify $B_n G$ with the set of maps

$$f : [n]^{(2)} = \{(i,j) \in [n] \times [n] : i \leq j\} \to G$$

which have the property that $f(i,i) = 0$ and $f(i,j) + f(j,k) = f(i,k)$ for $i \leq j \leq k$.

If $\mathcal{C}$ is a pointed ∞-category which admits finite colimits, then every $[n]$-gapped object $X : [n]^{(2)} \to \mathcal{C}$ determines a map $f : [n]^{(2)} \to K_0(\mathcal{C})$, given by $f(i,j) = [X(i,j)]$, and f will satisfy the condition above. This construction is functorial in $[n]$ and therefore gives rise to a map of simplicial spaces

$$S_\bullet(\mathcal{C}) \to B_\bullet(K_0(\mathcal{C})).$$

The natural map $K(\mathcal{C}) \to K_0(\mathcal{C})$ can then be obtained by passing to classifying spaces and then applying Ω. We may therefore rephrase Proposition 1 as follows:

Proposition 2. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0$ be a full stable subcategory such that every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{C}_0$. Then the diagram*

$$\begin{array}{ccc} |S_\bullet(\mathcal{C}_0)| & \longrightarrow & |S_\bullet(\mathcal{C})| \\ \downarrow & & \downarrow \\ |B_\bullet K_0(\mathcal{C}_0)| & \longrightarrow & |B_\bullet K_0(\mathcal{C})| \end{array}$$

is a homotopy pullback square.

Because the map $K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective and an object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if its K-theory class $[C]$ lifts to $K_0(\mathcal{C}_0)$, the diagram of simplicial spaces

$$\begin{array}{ccc} S_\bullet(\mathcal{C}_0) & \longrightarrow & S_\bullet(\mathcal{C}) \\ \downarrow & & \downarrow \\ B_\bullet K_0(\mathcal{C}_0) & \longrightarrow & B_\bullet K_0(\mathcal{C}) \end{array}$$

is a homotopy pullback square. As in the previous lecture, we need to show that this remains true after geometric realization. Once again, this conclusion is not purely formal, because the spaces $B_n K_0(\mathcal{C})$ are not connected (in fact, they are discrete). Our proof will proceed by taking advantage of some additional structure available in this situation: in this case, the coherently associative addition law on the spaces involved (given by the formation of coproducts in $\mathcal{C}$).

Notation 3. Let $\mathcal{S}$ denote the ∞-category of spaces and let Sp denote the ∞-category of spectra. The formation of 0th spaces determines a functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$. The ∞-category Sp is stable, so that products and coproducts coincide. Consequently, every object $E \in \mathrm{Sp}$ can be regarded as a commutative monoid object of Sp in an essentially unique way. It follows that Ω^∞ determines a map $\mathrm{Sp} \to \mathrm{CAlg}(\mathcal{S})$, where $\mathrm{CAlg}(\mathcal{S})$ denotes the ∞-category of commutative monoid objects of $\mathcal{S}$: that is, the ∞-category of E_∞-spaces.

It follows from abstract nonsense that the functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{CAlg}(\mathcal{S})$ admits a left adjoint, which we will denote by $X \mapsto X^{\mathrm{gp}}$. We will refer to X^{gp} as the *group completion* of X. Tautologically, any E_∞-space X is equipped with an E_∞-map $X \to \Omega^\infty X^{\mathrm{gp}}$. Nontautologically, one can show that this map is a homotopy equivalence if and only if X is grouplike: that is, $\pi_0 X$ is a group.

Let $\mathcal{C}$ be an ∞-category which admits finite coproducts. Then the formation of coproducts endows the underlying Kan complex $\mathcal{C}^{\simeq}$ with the structure of an E_∞-space. We will refer to the group completion $(\mathcal{C}^{\simeq})^{\mathrm{gp}}$ as the *additive K-theory spectrum* of $\mathcal{C}$ and denote it by $K_{\mathrm{add}}(\mathcal{C})$ (note that this conflicts with the notation of Lecture 14, where we used the same notation for the abelian group $\pi_0 K_{\mathrm{add}}(\mathcal{C})$).

If $\mathcal{C}$ is a pointed ∞-category which admits finite colimits, then each $\mathrm{Gap}_{[n]}(\mathcal{C})$ has the same property. It follows that each $S_n(\mathcal{C})$ is an E_∞-space which has a group completion $S_n(\mathcal{C})^{\mathrm{gp}}$. Since the geometric realization $|S_\bullet \mathcal{C}|$ is grouplike (it is connected, we have

$$\begin{aligned} |S_\bullet(\mathcal{C})| &\simeq \Omega^\infty(|S_\bullet(\mathcal{C})^{\mathrm{gp}}) \\ &\simeq \Omega^\infty(|S_\bullet(\mathcal{C})^{\mathrm{gp}}|). \end{aligned}$$

It follows that the diagram of Proposition 2 is obtained by applying Ω^∞ to a diagram of spectra

$$\begin{array}{ccc} |S_\bullet(\mathcal{C}_0)^{\mathrm{gp}}| & \longrightarrow & |S_\bullet(\mathcal{C})^{\mathrm{gp}}| \\ \downarrow & & \downarrow \\ |HB_\bullet K_0(\mathcal{C}_0)| & \longrightarrow & |HB_\bullet K_0(\mathcal{C})|. \end{array}$$

The functor Ω^∞ preserves pullback squares, and the formation of geometric realizations of spectra commutes with pullbacks (since the ∞-category Sp is stable). It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} S_n(\mathcal{C}_0)^{\mathrm{gp}} & \longrightarrow & S_n(\mathcal{C})^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ HB_n K_0(\mathcal{C}_0) & \longrightarrow & HB_n K_0(\mathcal{C}) \end{array}$$

is a pullback square. Replacing $\mathcal{C}$ by $\mathrm{Gap}_{[n]}(\mathcal{C})$, we can reduce to the case $n = 1$. Proposition 2 is now reduced to the following "additive" version:

Proposition 4. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0$ be a full stable subcategory such that every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{C}_0$. Then the diagram*

$$\begin{array}{ccc} K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & K_{\text{add}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ HK_0(\mathcal{C}_0) & \longrightarrow & HK_0(\mathcal{C}) \end{array}$$

is a homotopy pullback square.

Proof. This is a version of the group completion theorem. Let us indicate a proof. The spectra involved are connective, and the vertical maps are surjective on π_0. Consequently, it will suffice to show that the diagram of 0th spaces

$$\begin{array}{ccc} \Omega^\infty K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & \Omega^\infty K_{\text{add}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ K_0(\mathcal{C}_0) & \longrightarrow & K_0(\mathcal{C}) \end{array}$$

is a pullback square. Let Z denote the inverse image of $K_0(\mathcal{C}_0)$ in $\Omega^\infty K_{\text{add}}(\mathcal{C})$; we wish to show that the canonical map $\theta : \Omega^\infty K_{\text{add}}(\mathcal{C}_0) \to Z$ is a homotopy equivalence. Since $\Omega^\infty K_{\text{add}}(\mathcal{C}_0)$ and X are simple (they are infinite loop spaces), it will suffice to check that θ induces an isomorphism in homology.

Consider the singular chain complexes

$$A_0 = C_*(\Omega^\infty K_{\text{add}}(\mathcal{C}_0); \mathbf{Z}) \qquad A = C_*(\Omega^\infty K_{\text{add}}(\mathcal{C}); \mathbf{Z}).$$

Using the E_∞-structures on the spaces involved, we can regard A_0 and A as E_∞-algebras over $\mathbf{Z}$. Similarly, we have E_∞-algebras

$$B_0 = C_*(\mathcal{C}_0^{\simeq}; \mathbf{Z}) \qquad B = C_*(\mathcal{C}^{\simeq}; \mathbf{Z}).$$

Note that B contains B_0 as a direct summand, and in fact we have a natural grading $B = \bigoplus B_\alpha$ where α ranges over the cosets of $K_0(\mathcal{C}_0)$ in $K_0(\mathcal{C})$.

Using the universal property of the group completion, we see that A_0 can be obtained from B_0 by inverting all elements of the form $[X] \in \mathbf{Z}[K_0(\mathcal{C}_0)] \simeq \mathrm{H}_0(A_0)$ for $X \in \mathcal{C}_0$, and that A can be obtained from B by inverting all elements $[X]$ for $X \in \mathcal{C}$. However, since every object of $\mathcal{C}$ is a direct summand of an object in $\mathcal{C}_0$, we only need to invert the classes $[X]$ for $X \in \mathcal{C}_0$. We therefore have a canonical equivalence $A \simeq A_0 \otimes_{B_0} B$. This equivalence determines a direct sum decomposition

$$A \simeq \bigoplus_\alpha A_0 \otimes_{B_0} B_\alpha,$$

where the chain complex $C_*(X; \mathbf{Z})$ can be identified with the summand corresponding to $\alpha = 0$. From this description, it is clear that $A_0 \simeq C_*(X; \mathbf{Z})$. □

Sometimes there is not much difference between K-theory and additive K-theory. Roughly speaking, we would expect this behavior in a situation where every cofiber sequence

$$X' \to X \to X''$$

splits. However, this hypothesis is unreasonably strong in the context we have been discussing so far: for a cofiber sequence

$$X \to * \to \Sigma(X)$$

to split, we must have $X \simeq *$. It will therefore be useful to consider a slightly more general setup:

Definition 5. An ∞-*category with cofibrations* is a pointed ∞-category $\mathcal{C}$ with a distinguished class of morphisms, which we will call *cofibrations*, which satisfy the following axioms:

- All equivalences are cofibrations and the collection of cofibrations is closed under composition.
- For every object X in $\mathcal{C}$, the canonical map $* \to X$ is a cofibration.
- For a cofibration $f : X \to X'$ and an arbitrary map $X \to Y$, there exists a pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

  and the map g is also a cofibration.

Warning 6. We are using the term "cofibration" in order to follow the language of Waldhausen's paper, but the notion of cofibration considered above does not *a priori* have any relationship to the notion of cofibration in the language of model categories.

Example 7. Let $\mathcal{C}$ be a pointed ∞-category. One way to try to satisfy the axiomatics of Definition 5 is to have as many cofibrations as possible. We can make $\mathcal{C}$ into an ∞-category with cofibrations where *all* morphisms are cofibrations if and only if $\mathcal{C}$ has finite colimits.

Example 8. Let $\mathcal{C}$ be a pointed ∞-category. Another way to try to satisfy the axiomatics of Definition 5 is to have as few cofibrations as possible. Note that if for any pair of objects X and Y, the natural map $* \to X$ is a cofibration and therefore there exists a pushout square

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array}$$

where the lower horizontal map is a cofibration. Consequently, if $\mathcal{C}$ is an ∞-category with cofibrations, then $\mathcal{C}$ must have coproducts and every map of the form $Y \to X \vee Y$ must be a cofibration.

Conversely, suppose that $\mathcal{C}$ is a pointed ∞-category which admits finite coproducts. Then $\mathcal{C}$ can be made into an ∞-category with cofibrations by declaring that a morphism f is a cofibration if and only if it is equivalent to a morphism of the form $Y \to X \vee Y$; we will refer to such a morphism as a *split cofibration*.

Let $\mathcal{C}$ be an ∞-category with cofibrations. For each integer n, we let $\mathrm{Gap}_{[n]}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}\{(i,j) \in [n] \times [n] : i \leq j\}, \mathcal{C})$ spanned by those functors X satisfying the following three conditions:

- For each $i \leq j \leq k$, the natural map $X(i,j) \to X(i,k)$ is a cofibration.
- For each i, the object $X(i,i)$ is zero.
- For each $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} X(i,j) & \longrightarrow & X(i,k) \\ \downarrow & & \downarrow \\ * & \longrightarrow & X(j,k) \end{array}$$

  is a pushout square.

Arguing as in Lecture 16, we see that an object X of $\mathrm{Gap}_{[n]}(\mathcal{C})$ is determined by the diagram

$$X(0,1) \to X(0,2) \to \cdots \to X(0,n).$$

The only difference is that this time, we consider only those diagrams where each map is a cofibration.

Definition 9. Let $\mathcal{C}$ be an ∞-category with cofibrations. We let $S_\bullet(\mathcal{C})$ denote the simplicial space given by the formula $S_n(\mathcal{C}) = \mathrm{Gap}_{[n]}(\mathcal{C})^{\simeq}$, where $\mathrm{Gap}_{[n]}(\mathcal{C})$ is defined as above. We let $K(\mathcal{C})$ denote the space given by $\Omega|S_\bullet(\mathcal{C})|$.

The simplicial space $S_\bullet(\mathcal{C})$ and $K(\mathcal{C})$ depend not only on the ∞-category $\mathcal{C}$, but also on the class of cofibrations chosen. For example, if $\mathcal{C}$ admits finite colimits and we declare that all morphisms are cofibrations (Example 7), then we recover the definitions of Lecture 16. If $\mathcal{C}$ admits finite coproducts and we use only the split cofibrations (Example 8), then we often recover additive K-theory $K_{\mathrm{add}}(\mathcal{C})$:

Theorem 10. *Let $\mathcal{C}$ be an ∞-category which admits finite products and finite coproducts, and assume that the homotopy category of $\mathcal{C}$ is additive (so that finite products and finite coproducts in $\mathcal{C}$ coincide). For example, we can take any stable ∞-category, or any subcategory of a stable ∞-category which is closed under direct sums. Regard $\mathcal{C}$ as an ∞-category with cofibrations as in Example 8 (allowing only split cofibrations). Then there is a canonical homotopy equivalence $K_{\mathrm{add}}(\mathcal{C}) \to K(\mathcal{C})$ (where we abuse notation by identifying $K_{\mathrm{add}}(\mathcal{C})$ with its 0th space).*

We can identify the 0th space of $K_{\mathrm{add}}(\mathcal{C})$ with the $\Omega|Y_\bullet|$, where $Y_\bullet$ is the simplicial space given by $Y_n = (\mathcal{C}^{\simeq})^n$ (made into a simplicial space using the coproduct on $\mathcal{C}^{\simeq}$). The map $K_{\mathrm{add}}(\mathcal{C}) \to K(\mathcal{C})$ is then obtained from a map of simplicial spaces

$$Y_\bullet \to S_\bullet(\mathcal{C});$$

in degree n this map is given by the construction

$$(C_1, \ldots, C_n) \mapsto (C_1 \to C_1 \oplus C_2 \to \cdots \to C_1 \oplus \cdots \oplus C_n).$$

We wish to show that the induced map of geometric realizations $|Y_\bullet| \to |S_\bullet(\mathcal{C})|$ is a homotopy equivalence. All of the spaces in sight admit E_∞-structures coming from the formation of coproducts in $\mathcal{C}$. Arguing as before, we obtain homotopy equivalences

$$\begin{aligned} |Y_\bullet| &\simeq \Omega^\infty |Y_\bullet|^{\mathrm{gp}} \\ &\simeq \Omega^\infty |Y_\bullet^{\mathrm{gp}}|. \\ |S_\bullet(\mathcal{C})| &\simeq \Omega^\infty |S_\bullet(\mathcal{C})|^{\mathrm{gp}} \\ &\simeq \Omega^\infty |S_\bullet(\mathcal{C})^{\mathrm{gp}}|. \end{aligned}$$

It will therefore suffice to show that for each $n \geq 0$, the map $Y_n \to S_n(\mathcal{C})$ induces a homotopy equivalence of spectra $Y_n^{\mathrm{gp}} \to S_n(\mathcal{C})^{\mathrm{gp}}$.

For simplicity, let us consider the case $n = 2$ (the general case is only notationally more difficult). The space $S_2(\mathcal{C})$ classifies morphisms $f : X \to X'$ which are split cofibrations in $\mathcal{C}$. Let $e : S_2(\mathcal{C}) \to \mathcal{C}^{\simeq}$ be the map given by $e(X \to X') = X$, let $\pi : \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \to \mathcal{C}^{\simeq}$ be projection onto the first factor, and let $\iota : \mathcal{C}^{\simeq} \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ be given by $X \mapsto (*, X)$. We then have a commutative diagram of E_∞-spaces

$$\begin{array}{ccccc} \mathcal{C}^{\simeq} & \xrightarrow{\iota} & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} & \longrightarrow & S_2(\mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle e} \\ * & \longrightarrow & \mathcal{C}^{\simeq} & \xrightarrow{\mathrm{id}} & \mathcal{C}^{\simeq}. \end{array}$$

We wish to show that the upper left horizontal map becomes an equivalence after group completion. In other words, we wish to show that the square on the right becomes a pullback square after group completion. Since the ∞-category of spectra is stable, this is equivalent to the assertion that the square on the right becomes a pushout square after group completion. The left square is clearly a pushout after group completion; it will therefore suffice to show that the outer square is a pushout after group completion. In fact, we claim that the left square is a pushout *before* group completion. In other words, we claim that $\mathcal{C}^{\simeq}$ can be obtained as a one-sided bar construction $S_2(\mathcal{C}) \otimes_{\mathcal{C}^{\simeq}} *$ in the ∞-category of spaces, where the ∞-category $\mathcal{C}^{\simeq}$ acts on $S_2(\mathcal{C})$ via the construction

$$a : \mathcal{C}^{\simeq} \times S_2(\mathcal{C}) \to S_2(\mathcal{C})$$

$$(C, X \to X') \mapsto (X \to X' \oplus C).$$

This is an assertion which can be tested fiberwise over $\mathcal{C}^{\simeq}$. In other words, we are reduced to proving the following:

Proposition 11. *In the situation of Theorem 10, fix an object $X \in \mathcal{C}$, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}_{X/}$ spanned by the split cofibrations $X \to X'$. Let $\mathcal{C}^{\simeq}$ act on the space $\mathcal{D}^{\simeq}$ as above. Then the homotopy quotient*

$$\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq} := \mathcal{D}^{\simeq} \otimes_{\mathcal{C}^{\simeq}} *$$

is contractible.

Proof. Note that the ∞-category $\mathcal{D}$ admits finite coproducts (given by pushouts over X), so that $\mathcal{D}^{\simeq}$ is an E_∞-space. We can regard the quotient $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ as the cofiber of the natural map

$$f : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$$

$$C \mapsto (X \to X \oplus C)$$

in the ∞-category of E_∞-spaces. By construction, the map f is surjective on π_0 so that the quotient $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ is connected. In particular, $\mathcal{D}^{\simeq} / \mathcal{C}^{\simeq}$ is grouplike, so it can be identified with (the 0th space of) its group completion. It will therefore suffice to show that the map f induces an equivalence of group completions.

Define $q : \mathcal{D}^{\simeq} \to \mathcal{C}^{\simeq}$ by the formula $q(X \to X') = X'/X$. The map q is obviously a left homotopy inverse to f. To complete the proof, it will suffice to show that it is also a right homotopy inverse after group completion. In other words, we wish to show that the composite map

$$(f \circ q) : \mathcal{D}^{\simeq} \to \mathcal{D}^{\simeq}$$

$$(X \to X') \mapsto (X \to X \oplus (X'/X)$$

is homotopic to the identity map after group completion. In fact, we claim that $(f \circ q)$ is homotopic to the identity map id after adding a single copy of the identity map: that is, to any split cofibration $X \to X'$, we can functorially identify the split cofibrations

$$X \to X' \amalg_X X'$$

$$X \to X' \oplus (X'/X).$$

This identification follows from the additivity assumption on $\mathcal{C}$ (the "fold map" $X' \amalg_X X' \to X$ is split by the inclusion of either factor). □

References

[1] Waldhausen, F. *Algebraic K-theory of spaces.*