The Wall Finiteness Obstruction Revisited (Lecture 15)

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Let $\mathcal{C}$ be a (small) ∞-category which admits finite colimits. There is a formal procedure for enlarging $\mathcal{C}$ to admit all colimits: namely, one can replace $\mathcal{C}$ by the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-*objects* of $\mathcal{C}$. This enlargement can be characterized as follows:

(a) There is a fully faithful embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ (we will henceforth abuse notation by identifying $\mathcal{C}$ with its image in $\mathrm{Ind}(\mathcal{C})$).

(b) The ∞-category $\mathrm{Ind}(\mathcal{C})$ admits filtered colimits (in fact, it admits all small colimits, and the functor j preserves finite colimits).

(c) Every object of $\mathrm{Ind}(\mathcal{C})$ can be written as a filtered colimit $\varinjlim C_\alpha$, where each C_α belongs to $\mathcal{C}$ (identified with its image in $\mathrm{Ind}(\mathcal{C})$ via j).

(d) Every object $C \in \mathcal{C}$ is *compact* as an object of $\mathrm{Ind}(\mathcal{C})$: that is, the construction $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ commutes with filtered colimits.

At an informal level, we can see that these properties characterize $\mathrm{Ind}(\mathcal{C})$ as follows. For each filtered diagram $\{C_\alpha\}$ in the ∞-category $\mathcal{C}$, let "$\varinjlim C_\alpha$" denote the colimit of the diagram $\{C_\alpha\}$ in $\mathrm{Ind}(\mathcal{C})$. Assumption (b) ensures that this colimit is well defined and assumption (c) implies that every object of $\mathrm{Ind}(\mathcal{C})$ has this form. Morphism spaces in $\mathrm{Ind}(\mathcal{C})$ are then given by

$$\begin{aligned} \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\varinjlim C_\alpha\text{''}, \text{“}\varinjlim D_\beta\text{''}) &\simeq \varprojlim_\alpha \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, \text{“}\varinjlim D_\beta\text{''}) \\ &\simeq \varprojlim_\alpha \varinjlim_\beta \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, D_\beta) \\ &\simeq \varprojlim_\alpha \varinjlim_\beta \mathrm{Map}_{\mathcal{C}}(C_\alpha, D_\beta) \end{aligned}$$

where the last two equivalences are obtained from (d) and (a), respectively.

Remark 1. Working more formally, one can define $\mathrm{Ind}(\mathcal{C})$ to be the ∞-category of functors from $\mathcal{C}^{\mathrm{op}}$ to spaces which preserve finite limits, from which one can derive properties (a), (b), (c), and (d).

Example 2. Let $\mathcal{C}$ be the category of finitely presented groups. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the category of all groups. The same conclusion holds if we replace "groups" by any other type of algebraic structure (abelian groups, rings, commutative rings, etcetera).

Example 3. Let $\mathcal{C}$ be the ∞-category of finite CW complexes. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of spaces.

Example 4. Let R be a ring spectrum (or ordinary ring), and let $\mathcal{C}$ be the ∞-category of perfect (complexes of) R-modules. Then $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of all (complexes of) R-modules.

It is natural to ask if the converse to (d) is true: does every compact object of $\mathrm{Ind}(\mathcal{C})$ belong to (the essential image of) $\mathcal{C}$? To address, suppose we are given an object $X = \text{“}\varinjlim C''_\alpha \in \mathrm{Ind}(\mathcal{C})$ which is compact. Then the identity map $\mathrm{id}_X : X \to \text{“}\varinjlim C''_\alpha$ must factor through some C_α. It follows that X is a *retract* of C_α in the ∞-category $\mathrm{Ind}(\mathcal{C})$. Conversely, it is not hard to see that the collection of compact objects of $\mathrm{Ind}(\mathcal{C})$ is closed under retracts, and therefore contains all retracts of objects of $\mathcal{C}$.

Remark 5. Suppose that $X \in \mathrm{Ind}(\mathcal{C})$ is a retract of an object $C \in \mathcal{C}$, so that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & C & \end{array}$$

in $\mathcal{C}$. Let $r = \alpha \circ \beta$; we think of r as the “retraction” from C onto X. Then X can be recovered as the filtered colimit

$$C \xrightarrow{r} C \xrightarrow{r} C \xrightarrow{r} C \to \cdots$$

Conversely, if we start with a map $r : C \to C$ and define X to be the colimit of the above sequence, then one can show that X is a retract of C provided that there exists a homotopy h of r^2 with r for which the diagram

$$\begin{array}{ccc} r^3 & \xrightarrow{h\times\mathrm{id}} & r^2 \\ \downarrow{\scriptstyle \mathrm{id}\times h} & & \downarrow{\scriptstyle h} \\ r^2 & \xrightarrow{h} & r \end{array}$$

commutes up to homotopy. Beware that the commutativity of this diagram is a necessary condition: in general, not every idempotent in the homotopy category of $\mathcal{C}$ can be lifted to a “homotopy coherent” idempotent in $\mathcal{C}$.

Example 6. Let $\mathcal{C}$ be an ordinary category which admits finite colimits. If $X \in \mathrm{Ind}(\mathcal{C})$ is a retract of an object $C \in \mathcal{C}$, then X can be recovered as the coequalizer of the maps $r, \mathrm{id} : C \to C$. It follows that every compact object of $\mathrm{Ind}(\mathcal{C})$ belongs to (the essential image of) $\mathcal{C}$.

Example 7. Let $\mathcal{C}$ be the ∞-category of finite CW complexes, so that $\mathrm{Ind}(\mathcal{C})$ can be identified with the ∞-category of spaces. An object $X \in \mathrm{Ind}(\mathcal{C})$ is compact if and only if it is finitely dominated (in the sense of Lecture 2). Consequently, not all compact objects of $\mathrm{Ind}(\mathcal{C})$ come from $\mathcal{C}$.

Remark 8. Let $\mathcal{C}$ be a small ∞-category which admits finite colimits. We let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathrm{Ind}(\mathcal{C})$ spanned by the compact objects. Then $\overline{\mathcal{C}}$ is the *idempotent completion* of $\mathcal{C}$: that is, it can be obtained by formally enlarging $\mathcal{C}$ adding “images” of all coherently idempotent morphisms in $\mathcal{C}$.

Example 9. Let $\mathcal{C}$ be a stable ∞-category. Then the idempotent completion $\overline{\mathcal{C}}$ is also stable. Moreover, every object of $\overline{\mathcal{C}}$ can be obtained as a direct summand of an object of $\mathcal{C}$.

Example 10. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. The formation of idempotent completions commutes with filtered colimits. Applying this observation to the diagram

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots,$$

we conclude that the Spanier-Whitehead ∞-category $SW(\overline{\mathcal{C}})$ can be identified with the idempotent completion of $SW(\mathcal{C})$. In particular, every object of $SW(\overline{\mathcal{C}})$ can be written as a direct summand of an object of $SW(\mathcal{C})$.

In Lecture 2, we saw that not every finitely dominated space X is homotopy equivalent to a finite CW complex. However, the possible failure of X to be homotopy equivalent to a finite complex was under good control: namely, we could associate to X a certain algebraic invariant w_X (its *Wall finiteness obstruction*) which vanished if and only if X was homotopy equivalent to a finite CW complex. We would now like to do something like this for a general ∞-category $\mathcal{C}$. First, we need to introduce a bit of terminology.

Definition 11. Let $\mathcal{C}$ be an ∞-category which admits finite colimits. We will say that a morphism $f : X \to Y$ in $\mathrm{Ind}(\mathcal{C})$ is $\mathcal{C}$-*finite* if it can be written as a composition

$$X = X_0 \to X_1 \to \cdots \to X_n = Y$$

where each of the maps $X_{i-1} \to X_i$ fits into a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ X_{i-1} & \longrightarrow & X_i \end{array}$$

for $C, D \in \mathcal{C}$.

Example 12. Let $\mathcal{C}$ be the ∞-category of finite CW complexes. Then a map of spaces $f : X \to Y$ is $\mathcal{C}$-finite if and only if Y is homotopy equivalent to a space which is obtained from X by attaching finitely many cells.

Remark 13. The collection of $\mathcal{C}$-finite morphisms contains all equivalences and is closed under composition.

Remark 14. Let $\emptyset$ denote the initial object of $\mathcal{C}$. Then an object $X \in \mathrm{Ind}(\mathcal{C})$ belongs to $\mathcal{C}$ if and only if the map $\emptyset \to X$ is $\mathcal{C}$-finite.

Remark 15. Suppose we are given a pushout diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Ind}(\mathcal{C})$. If f' is $\mathcal{C}$-finite, then so is f.

We will also need the following less obvious observation:

Proposition 16. *Suppose we are given morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathrm{Ind}(\mathcal{C})$. *If f and $g \circ f$ are $\mathcal{C}$-finite, then g is $\mathcal{C}$-finite.*

Proof. Factoring f as a composition, we can reduce to the case where Y has the form $X \amalg_C D$ for some $C, D \in \mathcal{C}$. Then g factors as a composition

$$X \amalg_C D \to Z \amalg_C D \xrightarrow{\alpha} Z$$

where the first map is a pushout of $g \circ f$, and is therefore $\mathcal{C}$-finite. It will therefore suffice to show that α is $\mathcal{C}$-finite. This follows from the observation that α is a pushout of the "fold map" $D \amalg_C D \to D$. $\square$

Suppose now that X is an object of $\mathrm{Ind}(\mathcal{C})$. We let $\mathrm{Ind}(\mathcal{C})_{X//X}$ denote the ∞-category of objects of $\mathrm{Ind}(\mathcal{C})$ lying over and under X: that is, the ∞-category whose objects are commutative diagrams

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ X & & \xrightarrow{\mathrm{id}} & & X. \end{array}$$

Let $\mathcal{C}[X]$ denote the full subcategory of $\mathrm{Ind}(\mathcal{C})_{X//X}$ spanned by those diagrams where α is $\mathcal{C}$-finite. It is not hard to see that the objects of $\mathcal{C}[X]$ form compact generators for $\mathrm{Ind}(\mathcal{C})_{X//X}$: that is, we have an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{C}[X]) \simeq \mathrm{Ind}(\mathcal{C})_{X//X}$. In particular, we can identify the idempotent completion of $\mathcal{C}[X]$ with the ∞-category of compact objects of $\mathrm{Ind}(\mathcal{C})_{X//X}$ (which can be characterized as those diagrams above for which α exhibits Y as a compact object of $\mathrm{Ind}(\mathcal{C})_{X/}$).

Note that $\mathcal{C}[X]$ and its idempotent completion $\overline{\mathcal{C}[X]}$ are pointed ∞-categories which admit finite colimits, so that we can consider the K-groups

$$K_0(\mathcal{C}[X]) \qquad K_0(\overline{\mathcal{C}[X]})$$

introduced in the previous lecture. Note that if Y is a compact object of $\mathrm{Ind}(\mathcal{C})$ equipped with a map $f : Y \to X$, then the diagram

$$\begin{array}{ccccc} & & X \amalg Y & & \\ & \nearrow & & \overset{(\mathrm{id}, f)}{\searrow} & \\ X & & \xrightarrow{\mathrm{id}} & & X \end{array}$$

belongs to $\overline{\mathcal{C}[X]}$ and therefore determines an element of $K_0(\overline{\mathcal{C}[X]})$ which we will denote by $[Y/X]$. If Y belongs to $\mathcal{C}$, then the natural map $X \to X \amalg Y$ is $\mathcal{C}$-finite and we can therefore lift $[Y/X]$ to $K_0(\mathcal{C}[X])$. We are going to prove the following converse:

Theorem 17 (Generalized Wall Finiteness Obstruction)**.** *Let $\mathcal{C}$ be a small ∞-category which admits finite colimits and let $X \in \mathrm{Ind}(\mathcal{C})$. Then:*

- (a) *The natural map $\alpha : K_0(\mathcal{C}[X]) \to K_0(\overline{\mathcal{C}[X]})$ is a monomorphism of abelian groups.*
- (b) *Suppose that $X \in \mathrm{Ind}(\mathcal{C})$ is compact. Then X belongs to (the essential image of) $\mathcal{C}$ if and only if the class $[X/X] \in K_0(\overline{\mathcal{C}[X]})$ belongs to the image of α.*

Example 18. Suppose that $\mathcal{C}$ is the ∞-category of finite CW complexes and that X is a connected finitely dominated space. We will later show that the quotient $K_0(\widetilde{\mathcal{C}[X]})/K_0(\mathcal{C}[X])$ can be identified with the reduced K-group $\widetilde{K}_0(\mathbf{Z}[\pi_1 X])$ and that this identification carries $[X/X]$ to the Wall obstruction w_X. Consequently, assertion (b) of Theorem 17 can be regarded as a generalization of the main result of Lecture 2.

Proof of Theorem 17. Using Example 10, we can identify α with the canonical map from $K_0(SW(\mathcal{C}[X]))$ to $K_0(\overline{SW(\mathcal{C}[X])})$, which we proved to be injective in the previous lecture. This proves (a). The "only if" direction of (b) is obvious. We now prove the converse.

First, without loss of generality we can assume that X is the final object of $\mathrm{Ind}(\mathcal{C})$ (we can achieve this by replacing $\mathcal{C}$ by the ∞-category $\mathcal{C} \times_{\mathrm{Ind}(\mathcal{C})} \mathrm{Ind}(\mathcal{C})_{/X}$). Let us denote this final object by $*$. Let $\overline{\mathcal{C}}$ denote the idempotent completion of $\mathcal{C}$, so that $\overline{\mathcal{C}}$ contains $*$ as a final object. Then $\overline{\mathcal{C}[X]}$ can be identified with the ∞-category $\overline{\mathcal{C}}_*$ of pointed objects of $\overline{\mathcal{C}}$, and $\mathcal{C}[X]$ can be identified with the full subcategory of $\overline{\mathcal{C}}_*$ spanned by those pointed objects $e : * \to C$ where e is a $\mathcal{C}$-finite map.

To prove (b), we must show that if $[* \amalg *] \in K_0(\overline{\mathcal{C}}_*)$ belongs to the image of α, then the object $*$ belongs to $\mathcal{C}$. By assumption, we know that $*$ belongs to the idempotent completion of $\mathcal{C}$. In particular, there exists an object $C \in \mathcal{C}$ and a map $* \to C$. We then have a cofiber sequence

$$* \amalg * \to * \amalg C \to C$$

in the ∞-category $\overline{\mathcal{C}}_*$, where the middle term belongs to $\mathcal{C}[X]$. It follows that $[C] \in K_0(\widetilde{\mathcal{C}}_*)$ belongs to the image of α. Invoking the main result from the previous lecture, we conclude that the image of C in the Spanier-Whitehead category $SW(\widetilde{\mathcal{C}}_*)$ belongs to the Spanier-Whitehead category $SW(\mathcal{C}[X])$. In other words, there exists an integer $n \geq 0$ such that $\Sigma^n(C)$ belongs to $\mathcal{C}[X]$, meaning that the base point inclusion $* \to \Sigma^n(C)$ is $\mathcal{C}$-finite. Applying Proposition 16, we conclude that the projection map $\Sigma^n(C) \to *$ is $\mathcal{C}$-finite. If $n = 0$, this tells us that the map $C \to *$ is $\mathcal{C}$-finite so that $* \in \mathcal{C}$, as desired. We will complete the proof by verifying the following:

($*$) If $n \geq 0$ and there exists an object $C \in \mathcal{C}$ with a base point $* \to C$ for which the canonical map $\Sigma^{n+1}C \to *$ is $\mathcal{C}$-finite, then there is another object $D \in \mathcal{C}$ with a base point $* \to D$ for which the canonical map $\Sigma^n D \to *$ is also $\mathcal{C}$-finite.

(In fact, the proof will show that we can take D to be *any* other object of $\mathcal{C}[X]$, for example we can take $D = C$; however, it will be less confusing if we do not identify D with C.) To prove ($*$), we first note that because the ∞-category $\overline{\mathcal{C}}$ admits finite colimits it is naturally tensored over the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite CW complexes. We will denote the action of $\mathcal{S}^{\mathrm{fin}}$ on $\overline{\mathcal{C}}$ by

$$\otimes : \mathcal{S}^{\mathrm{fin}} \times \overline{\mathcal{C}} \to \overline{\mathcal{C}}.$$

Concretely, the tensor product $K \otimes C$ is characterized by the universal property $\mathrm{Map}_{\overline{\mathcal{C}}}(K \otimes C, D) = \mathrm{Map}_{\overline{\mathcal{C}}}(C, D)^K$. We will also abuse notation by identifying each finite CW complex K with the tensor product $K \otimes * \in \overline{\mathcal{C}}$.

The proof of ($*$) is based on the observation that the suspension $\Sigma^n C$ can be computed in two different ways:

(a) For any $n \geq 0$, we can identify $\Sigma^n C$ (for a pointed object C) with the pushout $(S^n \otimes C) \amalg_{S^n} *$.

(b) If $n > 0$, then the definition of $\Sigma^n C$ does not require a base point in C: the "unreduced suspension" can be realized as the pushout $C \amalg_{S^{n-1} \otimes C} S^{n-1}$

Let us now suppose that $C \in \mathcal{C}$ is as in ($*$) and use (b) to describe the suspension $\Sigma^{n+1}C$. Let D be any object of $\mathcal{C}$ equipped with a base point $* \to D$ (we know that such an object exists; for example we can take $D = C$). We then have a pushout diagram

$$\begin{array}{ccc} C \amalg_{S^n \otimes C} S^n & \longrightarrow & C \amalg_{S^n \otimes C} (S^n \otimes D) \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \otimes_{S^n} (S^n \otimes D). \end{array}$$

Since the left vertical map is $\mathcal{C}$-finite, so is the right vertical map. Since the upper right hand corner belongs to $\mathcal{C}$, we obtain $* \otimes_{S^n} (S^n \otimes D) \in \mathcal{C}$. Using description ($a$) of the suspension $\Sigma^n D$, we obtain a pushout diagram

$$\begin{array}{ccc} D & \longrightarrow & * \amalg_{S^n} (S^n \otimes D) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma^n D. \end{array}$$

The upper horizontal map in this diagram is a morphism between objects of $\mathcal{C}$ and is therefore $\mathcal{C}$-finite. It follows that the lower horizontal map is $\mathcal{C}$-finite again. Applying Proposition 16, we deduce that the projection map $\Sigma^n D \to *$ is $\mathcal{C}$-finite as desired. $\square$

References