(Lower) K-theory of ∞-categories (Lecture 14)

October 2, 2014

We begin by reviewing the definition of an ∞-category.

Notation 1. For every pair of integers $0 \leq i \leq n$, we let Λ^n_i denote the simplicial subset of Δ^n given by the union of all those faces except the one opposite to the ith vertex. We will refer to a simplicial set of the form Λ^n_i as a *horn*. We will say that it is an *inner horn* if $0 < i < n$, and otherwise an *outer horn*.

Definition 2. Let X be a simplicial set. We will say that X is an ∞-*category* if every map $f_0 : \Lambda^n_i \to X$ can be extended to an n-simplex $f : \Delta^n \to X$ provided that $0 < i < n$. (In other words, every *inner horn* in X can be filled.)

Remark 3. Simplicial sets satisfying the requirements of Definition 2 are also referred to as *quasi-categories* or *weak Kan complexes*.

Example 4. Any Kan complex is an ∞-category (recall that a simplicial set X is a Kan complex if *any* horn $f_0 : \Lambda^n_i \to X$ can be extended to an n-simplex of X).

Example 5. For any category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.

In fact, one has the following stronger assertion:

Exercise 6. Let X be a simplicial set. Show that X is isomorphic to the nerve of a category if and only if every inner horn $f_0 : \Lambda^n_i \to X$ can be extended *uniquely* to an n-simplex $f : \Delta^n \to X$.

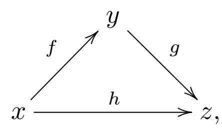
In what follows, we will often abuse notation by identifying a category $\mathcal{C}$ with the ∞-category $\mathrm{N}(\mathcal{C})$. This does not lose any information:

Exercise 7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there is a bijective correspondence between the set of functors $F : \mathcal{C} \to \mathcal{D}$ and the set of maps of simplicial sets $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$. In other words, the formation of nerves induces a fully faithful embedding from the category of (small) categories to the category of simplicial sets.

The formation of nerves admits a left adjoint, which sends each simplicial set X to a category which we will denote by hX. Concretely, the category hX admits the following presentation by generators and relations:

- The objects of hX are the vertices of X.
- For each edge e of X joining a vertex x to a vertex y, there is a corresponding morphism $[e]$ from x to y in hX.
- If the edge e is degenerate (so that $x = y$), then $[e] = \mathrm{id}_x$.

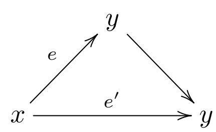
- For every 2-simplex of X given pictorially by the diagram



 we have $[h] = [g] \circ [f]$ in hX.

We will refer to hX as the *homotopy category* of X.

In the special case where X is an ∞-category, the homotopy category hX admits a more concrete description: all morphisms in hX have the form $[e]$ for some edge e, and two edges e and e' (with the same initial and final vertices) satisfy $[e] = [e']$ if and only if they are homotopic (meaning that there exists a 2-simplex



whose 0th face (joining y to y) is degenerate).

Most of the basic concepts of category theory (commutative diagrams, limits and colimits, initial and final objects, functors, adjunctions) can be generalized to the setting of ∞-categories. We will henceforth make use of those generalizations, and refer the reader to [1] for more details.

In what follows, we will use the notation $\mathcal{C}$ to denote an ∞-category (emphasizing the idea that $\mathcal{C}$ is some sort of generalized category rather than a simplicial set). We refer to the vertices of $\mathcal{C}$ as its *objects* and to the edges of $\mathcal{C}$ as its *morphisms*.

Definition 8. Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object $*$ which is both initial and final. We will say that $\mathcal{C}$ is *pointed* if it has a zero object. If $\mathcal{C}$ is pointed, then for every pair of objects X and Y there is a canonical morphism from X to Y given by the composition $X \to * \to Y$, which we refer to as the *zero morphism*.

Notation 9. Let $\mathcal{C}$ be a pointed ∞-category with zero object $*$. Suppose that $\mathcal{C}$ admits pushouts. For every morphism $f : X \to Y$ in $\mathcal{C}$, we let $\mathrm{cofib}(f)$ denote the pushout $Y \amalg_X *$. We refer to f as the *cofiber* of f. In the special case where $Y = *$, we refer to $\mathrm{cofib}(f)$ as the *suspension* of X and denote it by ΣX. Note that we have a diagram

$$X \xrightarrow{f} Y \to \mathrm{cofib}(f)$$

where the composition is zero; we refer to such diagrams as *cofiber sequences*.

Definition 10. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. We let $K_0(\mathcal{C})$ denote the free abelian group on generators $[X]$, where X is an object of $\mathcal{C}$, modulo the relations given by $[X'] + [X''] = [X]$ whenever there is a cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$.

Remark 11. Using the cofiber sequence

$$* \to * \to *$$

we deduce that $[*] = 0 \in K_0(\mathcal{C})$. Using the cofiber sequence

$$X \to * \to \Sigma(X)$$

we conclude that $[\Sigma(X)] = -[X]$ in $K_0(\mathcal{C})$.

Warning 12. Definition 10 is not interesting for "large" ∞-categories. For example, if $\mathcal{C}$ admits infinite coproducts, then any object X fits into a cofiber sequence

$$\coprod_{n \geq 1} X \to \coprod_{n \geq 0} X \to X$$

where the first two terms are equivalent to one another, so that $[X] = 0 \in K_0(\mathcal{C})$; since X was arbitrary, we have $K_0(\mathcal{C}) \simeq 0$.

Example 13. Let $\mathcal{C}$ be the ∞-category of finite pointed spaces. Then $K_0(\mathcal{C})$ is isomorphic to $\mathbf{Z}$, the isomorphism being given by the "reduced" Euler characteristic

$$[X] \mapsto \chi_{\text{hyp}}(X) = \chi(X) - 1.$$

Example 14. Let R be a ring. A *perfect complex* over R is a bounded chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

where each P_i is a finitely generated projective R-module. The collection of perfect chain complexes over R can be organized into an ∞-category $\text{Mod}_R^{\text{perf}}$. There is a natural map $K_0(R) \to K_0(\text{Mod}_R^{\text{perf}})$ which carries each finitely generated projective R-module P to the chain complex consisting of P in degree zero. One can show that this map is an isomorphism: it has an inverse which carries a chain complex $[P_*]$ to the alternating sum $\sum_n (-1)^n [P_n]$.

Remark 15. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit pushouts and suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves finite colimits. Then F induces a group homomorphism $K_0(\mathcal{C}) \to K_0(\mathcal{D})$, given by $[X] \mapsto [F(X)]$.

Example 16. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. Then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ satisfies the hypotheses of Remark 15, and induces the map $K_0(\mathcal{C}) \to K_0(\mathcal{C})$ given by multiplication by -1.

Definition 17. We say that an ∞-category $\mathcal{C}$ is *stable* if it is pointed, admits pushouts, and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories.

Remark 18. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. We define the *Spanier-Whitehead category* $SW(\mathcal{C})$ to be the direct limit

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots .$$

Then $SW(\mathcal{C})$ is stable, and is universal among stable ∞-category which receive a functor from $\mathcal{C}$ which preserves finite colimits. Moreover, $K_0(SW(\mathcal{C}))$ can be identified with the direct limit of the sequence

$$K_0(\mathcal{C}) \xrightarrow{-1} K_0(\mathcal{C}) \xrightarrow{-1} K_0(\mathcal{C}) \to \cdots ,$$

and is therefore isomorphic to $K_0(\mathcal{C})$.

In other words, for studying K_0, there is no real loss of generality in assuming that we are working with stable ∞-categories.

In the next lecture, we will need the following result:

Proposition 19. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full (stable) subcategory. Assume that every object of $\mathcal{C}$ is a direct summand of an object that belongs to $\mathcal{C}_0$. Then:*

(*a*) *The canonical map $\alpha : K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective.*

(*b*) *An object $C \in \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if $[C]$ belongs to the image of α.*

To prove Proposition 19, it will be convenient to introduce a variant of Definition 10.

Definition 20. Let $\mathcal{C}$ be a stable ∞-category. We let $K_{\text{add}}(\mathcal{C})$ denote the free abelian group generated by symbols $[X]$ where $X \in \mathcal{C}$, modulo the relations

$$[X] = [X'] + [X'']$$

if X is equivalent to a direct sum $X' \oplus X''$.

Remark 21. In the situation of Definition 20, it is easy to see that we have $[X] = [Y]$ in $K_{\text{add}}(\mathcal{C})$ if and only if X and Y are *stably equivalent*: that is, if and only if there exists an object $Z \in \mathcal{C}$ such that $X \oplus Z$ is equivalent to $Y \oplus Z$.

We have an evident surjective map $K_{\text{add}}(\mathcal{C}) \to K_0(\mathcal{C})$; let us denote the kernel of this map by $I(\mathcal{C})$.

Lemma 22. *In the situation of Proposition 19, the canonical map $I(\mathcal{C}_0) \to I(\mathcal{C})$ is surjective.*

Proof. Note that $I(\mathcal{C})$ is generated by expressions of the form $\eta = [X] - [X'] - [X'']$, where

$$X' \to X \to X''$$

is a cofiber sequence in $\mathcal{C}$. For any such cofiber sequence, we can choose objects Y', Y'' in $\mathcal{C}$ such that $X' \oplus Y'$ and $X'' \oplus Y''$ belong to $\mathcal{C}_0$. We then have a cofiber sequence

$$X' \oplus Y' \to X \oplus Y' \oplus Y'' \to X'' \oplus Y''$$

where the outer terms belong to $\mathcal{C}_0$, so that the middle term does as well. It follows that $\eta = [X \oplus Y' \oplus Y''] - [X' \oplus Y'] - [X'' \oplus Y'']$ belongs to the image of the map $I(\mathcal{C}_0) \to I(\mathcal{C})$. □

Proof of Proposition 19. It follows immediately from Remark 21 that the map $K_{\text{add}}(\mathcal{C}_0) \to K_{\text{add}}(\mathcal{C})$ is injective. Assertion (a) now follows by applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \to & I(\mathcal{C}_0) & \to & K_{\text{add}}(\mathcal{C}_0) & \to & K_0(\mathcal{C}_0) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I(\mathcal{C}) & \to & K_{\text{add}}(\mathcal{C}) & \to & K_0(\mathcal{C}) & \to & 0. \end{array}$$

To prove (b), we note that the snake lemma implies that the natural map

$$K_{\text{add}}(\mathcal{C}) / \operatorname{Im}(K_{\text{add}}(\mathcal{C}_0)) \to K_0(\mathcal{C}) / \operatorname{Im} K_0(\mathcal{C}_0)$$

is an isomorphism of abelian groups. Consequently, if $X \in \mathcal{C}$ has the property that $[X]$ belongs to the image of $K_0(\mathcal{C}_0)$, then $[X]$ belongs to the image of $K_{\text{add}}(\mathcal{C}_0)$. It follows that there exists objects $Y, Y' \in \mathcal{C}_0$ such that $X \oplus Y \simeq Y'$, so that X is equivalent to the cofiber of a map $Y \to Y'$ and therefore belongs to $\mathcal{C}_0$ as desired. □

References

[1] Lurie, J. *Higher Topos Theory.*