Homotopy Types and Simple Homotopy Types (Lecture 13)

September 30, 2014

In the first part of this course, we introduced a simplicial set $\mathcal{M}$ whose n-simplices are finite polyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^n$ is a fibration. We can think of $\mathcal{M}$ as a "moduli space of simple homotopy types": we have seen that two polyhedra belong to the same path component of $\mathcal{M}$ if and only if they are simple homotopy equivalent.

Let us now consider a variant of the simplicial set $\mathcal{M}$ where all finiteness conditions are replaced by homotopy-theoretic finiteness conditions.

Definition 1. Let Q be the Hilbert cube (or some other suitably enormous contractible space). We let $\mathcal{M}^h$ be the simplicial set whose n-simplices are subspaces $E \subseteq \Delta^n \times Q$ for which the projection map $E \to \Delta^n$ is a fibration whose fibers are finitely dominated.

Remark 2. The simplicial set $\mathcal{M}^h$ is a Kan complex. For any simplicial set B, the set of homotopy classes of maps from B to $\mathcal{M}^h$ can be identified with the set of homotopy equivalence classes of fibrations $E \to B$ (in the category of simplicial sets) or $E \to |B|$ (in the category of topological spaces) whose fibers are finitely dominated.

Remark 3. The simplicial set $\mathcal{M}^h$ is homotopy equivalent to the disjoint union $\coprod_X \mathrm{BAut}(X)$, where X ranges over all homotopy equivalence classes of finitely dominated spaces and $\mathrm{Aut}(X)$ denotes the simplicial monoid of homotopy equivalences of X with itself.

A choice of inclusion $\mathbb{R}^\infty \hookrightarrow Q$ induces a map of simplicial sets $\mathcal{M} \to \mathcal{M}^h$, which we think of as assigning to each simple homotopy type its underlying homotopy type.

The second part of this course is devoted to the following question:

Question 4. What can one say about the homotopy fibers of the map $\mathcal{M} \to \mathcal{M}^h$?

For every finitely dominated space X, let $\mathfrak{S}(X)$ denote the homotopy fiber product $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$. We can identify the connected components of $\mathfrak{S}(X)$ with equivalence classes of homotopy equivalences $f : X \to P$ where P is a finite polyhedron, where we declare $f : X \to P$ to be equivalent to $f' : X \to P'$ if the induced homotopy equivalence of P with P' is simple. If X is connected with fundamental group G, then the first few lectures yield the following information:

- The space $\mathfrak{S}(X)$ is nonempty if and only if the Wall finiteness obstruction of X vanishes (as an element of $\widetilde{K}_0(\mathbf{Z}[G])$).
- If $\mathfrak{S}(X)$ is nonempty, then it is a torsor for the Whitehead group $\mathrm{Wh}(G) = \widetilde{K}_1(\mathbf{Z}[G])/\operatorname{Im}(G)$.

In particular, if X is a finite polyhedron to begin with (so that $\mathfrak{S}(X)$ has a canonical base point), then we obtain an abelian group structure on the set $\pi_0\,\mathfrak{S}(X)$. We next observe that this is no accident.

Let X be a finite polyhedron. We can identify $\mathfrak{S}(X)$ with the simplicial set whose n-simplices are pairs (E, ϕ), where $E \subseteq \Delta^n \times \mathbb{R}^\infty$ is a finite polyhedron for which the projection $E \to \Delta^n$ is a fibration, and $\phi : X \times \Delta^n \to E$ is a PL homotopy equivalence which commutes with the projection to Δ^n. As usual, we will generally abuse terminology by ignoring the data of the embedding $E \hookrightarrow \Delta^n \times \mathbb{R}^\infty$ when describing our constructions.

Let $\mathfrak{S}'(X)$ denote the simplicial subset of $\mathfrak{S}(X)$ consisting of those pairs (E, ϕ) where ϕ is an embedding. It is not hard to see that $\mathfrak{S}'(X)$ is a deformation retract of $\mathfrak{S}(X)$: we can always replace a map $\phi : X \times \Delta^n \to E$ with the diagonal map $(\phi, j) : X \times \Delta^n \to E \times C(X)$, where $C(X)$ is some contractible polyhedron equipped with an embedding $j : X \to C(X)$.

There is a natural "addition" map on the Kan complex $\mathfrak{S}'(X)$: on n-simplices, it carries a pair of embeddings

$$\phi : X \times \Delta^n \hookrightarrow E \qquad \phi' : X \times \Delta^n \hookrightarrow E'$$

to the induced map $X \times \Delta^n \to E \amalg_{X \times \Delta^n} E'$. This addition is commutative and associative up to coherent homotopy, and therefore endows $\mathfrak{S}'(X) \simeq \mathfrak{S}(X)$ with the structure of an E_∞-space.

At the level of vertices, we note that if we are given embeddings $\phi : X \to E$ and $\phi' : X \to E'$, then the Whitehead torsion of the induced map $X \hookrightarrow E \amalg_X E'$ is the sum $\tau(\phi) + \tau(\phi')$. In other words, the commutative monoid structure on $\pi_0 \, \mathfrak{S}(X)$ determined by the E_∞-structure on $\mathfrak{S}(X)$ agrees with the abelian group structure arising from the identification $\pi_0 \, \mathfrak{S}(X) \simeq \mathrm{Wh}(G)$. Consequently, the E_∞-structure on $\mathfrak{S}(X)$ is group-like, and therefore exhibits $\mathfrak{S}(X)$ as the zeroth space of some spectrum. Our goal in the second part of this course will be to give a more explicit description of this spectrum.

Recall that if X is any space for which the homology $\mathrm{H}_*(X; \mathbf{Q})$ is finite-dimensional as a rational vector space (for example, any finitely dominated space), then the *Euler characteristic* of X is given by the alternating sum

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}_i(X; \mathbf{Q}).$$

This is manifestly a homotopy invariant quantity. However, it can be described in other ways which are not so obviously invariant:

Example 5. Let X be a finite CW complex. Then we have Euler's formula

$$\chi(X) = \sum_{d \geq 0} (-1)^d s_d,$$

where s_d denotes the number of d-cells of X.

Example 6. If X be a compact Riemannian manifold of even dimension $2n$. Then the Chern-Gauss-Bonnet formula gives

$$\chi(X) = \frac{1}{(2\pi)^n} \int_X \Omega$$

where Ω denotes the Pfaffian of the curvature of X.

In both of these examples, we took advantage of some additional structure on X (a CW structure or a Riemannian structure) to write the homotopy invariant quantity $\chi(X)$ as a "sum" of local contributions which depend on that additional structure. In either case, the additional structure on X that we needed equips X with a preferred simple homotopy type. The next main theorem of this course will be a sort of converse: we will show that if X is a finitely dominated space, then having a "local formula" for the Euler characteristic of X (in a suitably generalized setting) is equivalent to equipping X with a preferred simple homotopy type.

Let us now be a bit more precise. A fundamental property of the Euler characteristic is that it is additive for homotopy pushout squares: that is, given a homotopy pushout square of finitely dominated spaces

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

where the horizontal maps are inclusions of subcomplexes, we have $\chi(Y') + \chi(X) = \chi(X') + \chi(Y)$.

Suppose, more generally, that we are given some "target space" Z. Let $T(Z)$ denote the free abelian group generated by symbols $[X]$, where X is a finitely dominated space with a map to Z, modulo the relations given by

$$[\emptyset] = 0$$

$$[Y'] + [X] = [X'] + [Y]$$

for every homotopy pushout diagram of CW complexes over Z, as above. For every finitely dominated space X with a map $X \to Z$, we can regard $[X]$ as a kind of "generalized Euler characteristic" of X, which is not an integer but an element of the abelian group $T(Z)$.

In the next few lectures, we will see that the abelian group $T(Z)$ can be identified with π_0 of a spectrum $A(Z)$, called the *Waldhausen A-theory spectrum of* Z.

Warning 7. Our definition of $A(Z)$ will differ slightly from the definition more common in the literature, in that we will allow finitely generated projective modules rather than only finitely generated free modules; this has the effect of slightly enlarging the group π_0 and not changing the higher homotopy groups.

The construction $Z \mapsto A(Z)$ is functorial. Consequently, any point $z \in Z$ determines a map $A(*) \to A(Z)$. This map depends continuously on Z, and we therefore obtain an *assembly map*

$$\rho : A(*) \wedge Z_+ \to A(Z).$$

Now suppose that the space Z itself is finitely dominated, so that $[Z]$ can be regarded as element of the abelian group $\pi_0 A(Z)$ (which we will identify with a point of $\Omega^\infty A(Z)$. This point can be regarded as a "universal" version of the Euler characteristic of Z, and the homotopy fiber of the map

$$\Omega^\infty(A(*) \wedge Z_+) \to \Omega^\infty A(Z)$$

can be thought of as the "space of all local formulas for the Euler characteristic of Z." We can now state more precisely the theorem we are after:

Theorem 8. *Let Z be a finitely dominated space. Then there is a canonical homotopy pullback square*

$$\begin{array}{ccc} \mathfrak{S}(Z) & \longrightarrow & \Omega^\infty(A(*) \wedge Z_+) \\ \downarrow & & \downarrow \\ * & \xrightarrow{[Z]} & \Omega^\infty A(Z). \end{array}$$

Theorem 8 motivates the following:

Definition 9. Let Z be a space. The *Whitehead spectrum* of Z is the cofiber of the assembly map $A(*) \wedge Z_+ \to A(Z)$. We will denote the Whitehead spectrum of Z by $\mathbf{Wh}(Z)$.

Theorem 8 implies that if Z is a finitely dominated space, then we can associate to Z an obstruction $\eta \in \pi_0 \mathbf{Wh}(Z)$ which vanishes if and only if $\mathfrak{S}(Z)$ is nonempty. If $\eta = 0$, then $\mathfrak{S}(Z)$ is homotopy equivalent to the space $\Omega^{\infty+1}\mathbf{Wh}(Z)$.

We will see that if Z is connected with fundamental group G, then there are canonical isomorphisms

$$\pi_0 \mathbf{Wh}(Z) \simeq \widetilde{K}_0(\mathbf{Z}[G])$$

$$\pi_1 \mathbf{Wh}(Z) \simeq \mathrm{Wh}(G).$$

Consequently, Theorem 8 will subsume the theory of the Wall finiteness obstruction and the theory of the Whitehead torsion.

Warning 10. The groups $\pi_0 \mathbf{Wh}(Z)$ and $\pi_1 \mathbf{Wh}(Z)$ depend only on the fundamental group of Z (if Z is connected), but this is not true in general: the Whitehead spectrum $\mathbf{Wh}(Z)$ is sensitive to the entire homotopy type of the space Z.