Equivalence of the Combinatorial Definition (Lecture 11)

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Our goal in this lecture is to complete the proof of our first main theorem by proving the following:

Theorem 1. *The map of simplicial sets*

$$\gamma : \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}) \to \mathcal{M}$$

constructed in the previous lecture is a weak homotopy equivalence.

Here $\mathcal{C}_{\mathrm{cof}}$ denotes the category of finite partially ordered sets and left cofinal maps.

Recall that the simplicial set $\mathcal{M}$ is characterized by the following property: for any finite nonsingular simplicial set X, maps from X into $\mathcal{M}$ can be identified with finite polyhedra $E \subseteq |X| \times \mathbb{R}^\infty$ for which the projection map $E \to |X|$ is a fibration. In what follows, we will abuse notation by ignoring the embeddings into $\mathbb{R}^\infty$ and simply identifying maps $X \to \mathcal{M}$ with the corresponding PL fibration $E \to |X|$.

The first observation is that γ is surjective on connected components. To prove this, it suffices to observe that every finite polyhedron X is PL homeomorphic to the nerve of a finite partially ordered set P. In fact, we can take P to be the set $\Sigma(X)$ of simplices taken with respect to some chosen triangulation τ of X (so that the canonical triangulation of $\mathrm{N}(P)$ can be identified with the barycentric subdivision of the triangulation τ).

The preceding observation can be carried out in families. Let us first introduce a bit of terminology.

Definition 2. Let us say that a Cartesian fibration of partially ordered sets $f : P \to Q$ is *good* if, for every inequality $q \leq q'$ in Q, the induced map $P_{q'} \to P_q$ is left cofinal.

Remark 3. Let $f : P \to Q$ be a Cartesian fibration finite partially ordered sets. The following conditions are equivalent:

- (i) The map f is good.
- (ii) For every $p \in P$ and every $q \geq f(p)$ in Q, the poset $\{a \in P : f(a) = q \text{ and } a \geq p\}$ is weakly contractible.
- (iii) The induced map of topological spaces $|\mathrm{N}(P)| \to |\mathrm{N}(Q)|$ is a fibration.

Remark 4. Let $f : P \to Q$ be a good Cartesian fibration of finite partially ordered sets. Then the construction $(q \in Q) \mapsto P_q$ determines a functor $Q \to \mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}$, hence a map of simplicial sets $\chi_f : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. The composite map

$$\mathrm{N}(Q) \xrightarrow{\chi_f} \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}) \xrightarrow{\gamma} \mathcal{M}$$

classifies the PL fibration $|\mathrm{N}(P)| \to |\mathrm{N}(Q)|$.

Construction 5. Let $q : E \to B$ be a PL fibration of polyhedra. Suppose that we have chosen compatible triangulations of E and B, and let $\Sigma(E)$ and $\Sigma(B)$ denote the posets of simplices of E and B, respectively. Since q is a fibration, the map $\Sigma(E) \to \Sigma(B)$ is good; we therefore obtain a map $\chi_q : \mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$, whose composition with γ classifies the PL fibration $|\mathrm{N}(\Sigma(E))| \to |\mathrm{N}(\Sigma(B))|$ which is isomorphic (in the category of polyhedra) to our original fibration q.

Note that the map χ_q of Construction 5 depends not only on q, but also on our chosen triangulations of E and B. In fact, even the domain of the map χ_q depends on the choice of triangulation. However, we would like to argue that this dependence is not essential. For this, we introduce the following definition:

Definition 6. Let X be a simplicial set and suppose that we are given a pair of maps $f : A \to X$, $f' : A' \to X$. We will say that f and f' are *homotopic* if there exists another map $\overline{f} : \overline{A} \to X$ and trivial cofibrations equivalences $i : A \to \overline{A}$ and $i' : A' \to \overline{A}$ such that $f = \overline{f} \circ i$ and $f' = \overline{f} \circ i'$. (In this case, we will refer to f' as a *homotopy* from f to f').

The proof of Theorem 1 rests on the following three propositions. First, we claim that Construction 5 is not very sensitive to our chosen triangulations:

Proposition 7. *Given diagram of triangulated polyhedra*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ B' & \longrightarrow & B \end{array}$$

where the vertical maps are fibrations which are compatible with our triangulations and the horizontal maps are PL homeomorphisms, the maps $\chi_q : \mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ *and* $\chi_{q'} : \mathrm{N}(\Sigma(B')) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ *are homotopic to one another.*

We next claim that Construction 5 is homotopy inverse to γ:

Proposition 8. *Let $P \to Q$ be a good Cartesian fibration of finite partially ordered sets, classified by a map* $e : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ *so that $\gamma(e)$ classifies the PL fibration* $q : |\,\mathrm{N}(P)| \to |\,\mathrm{N}(Q)|$. *Applying Construction 5 to the standard triangulations of* $\mathrm{N}(P)$ *and* $\mathrm{N}(Q)$, *we obtain a map* $\chi_q : \mathrm{N}(\Sigma(\mathrm{N}(Q))) = \mathrm{N}(\mathrm{Chain}(Q)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. *Then e and χ_q are homotopic.*

To state our next result, we need a bit of notation. For every finite polyhedron X, we let $[X]$ denote the associated vertex of $\mathcal{M}$. We will be a bit sloppy and identify $[X]$ with $[Y]$ when we are given a PL homeomorphism from X to Y (recall that to fully specify a vertex of $\mathcal{M}$, we need to supply an embedding of the corresponding polyhedron into $\mathbb{R}^\infty$, but we are suppressing this). Similarly, if P is a finite partially ordered set, we let $[P]$ denote the corresponding vertex of $\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$, so that $\gamma([P]) = [\mathrm{N}(P)]$.

If E is a finite polyhedron equipped with a triangulation, then Construction **??** (applied in the case $B = *$) determines a map $* \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ which is given by the vertex $[\Sigma(E)]$. If we are given another triangulation of the same polyhedron (which we will denote by E' to avoid confusion), then Proposition 7 determines a path p from $[\Sigma(E)]$ to $[\Sigma(E')]$ in the simplicial set $\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. The image of this path under γ is a loop based at $[E] = [E']$.

Proposition 9. *In the situation above, the homotopy of Proposition 7 can be chosen so that $\gamma(p)$ is homotopic to the constant loop in $\mathcal{M}$.*

Let us now show that Propositions 7, 8, and 9 imply Theorem 1.

Proof of Theorem 1. Fix a finite polyhedron X and a triangulation of X. We will prove the following for each $n \geq 0$:

(*a*) If $\eta \in \pi_n(|\,\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|, [\Sigma(X)])$ has the property that $\gamma(\eta)$ vanishes in $\pi_n(|\,\mathcal{M}\,|, [X])$, then e vanishes.

(*b*) Every element of $\pi_n(|\,\mathcal{M}\,|, [X])$ has the form $\gamma(\eta)$ for some $\eta \in \pi_n(|\,\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|, [\Sigma(X)])$.

Note that when $n = 0$, condition (a) asserts that the connected component of $[\Sigma(X)]$ in $|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})|$ is the unique connected component which lies over the connected component of $[X]$ in $|\mathcal{M}|$. Allowing X to vary, it follows that the map $\pi_0|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})| \to \pi_0|\mathcal{M}|$ is bijective. For $n > 0$, condition (a) implies that the map of homotopy groups

$$\pi_n(|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})|, [\Sigma(X)]) \to \pi_n(|\mathcal{M}|, [X])$$

is injective and condition (b) implies that it is surjective. Theorem 1 will then follow from Whitehead's theorem.

Let us begin with the proof of (a). Suppose we are given a homotopy class $\eta \in \pi_n(|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})|, [\Sigma(X)])$ such that $\gamma(\eta)$ vanishes. The simplicial set $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ is not a Kan complex, so the class η cannot necessarily be represented by a map of simplicial sets from $\partial\,\Delta^{n+1}$ into $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. However, it can always be represented by a map of simplicial sets after subdividing $\partial\,\Delta^{n+1}$ sufficiently many times. We may therefore assume that η is represented by a map

$$f : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}),$$

where Q is some finite partially ordered set such that $|\mathrm{N}(Q)|$ is homeomorphic to S^n. The data of the map f is equivalent to the data of a functor $Q \to \mathcal{C}_{\mathrm{cof}}^{\mathrm{op}}$, which determines a PL fibration $q : |\mathrm{N}(P)| \to |\mathrm{N}(Q)|$.

Since $\gamma(\eta)$ vanishes, we can extend q to a PL fibration $\overline{q} : E \to B$ where $B \simeq D^{n+1}$ is contractible. Let us choose compatible triangulations of E and B, so that we have partially ordered sets of simplices $\Sigma(E)$ and $\Sigma(B)$. Refining these triangulations if necessary, we may assume that the images of $|\mathrm{N}(P)|$ and $|\mathrm{N}(Q)|$ are subcomplexes $E' \subseteq E$ and $B' \subseteq B$, and that the triangulations on E' and B' refine the triangulations of $\mathrm{N}(P)$ and $\mathrm{N}(Q)$, respectively. Construction 5 then yields a map of simplicial sets $\chi_{\overline{q}} : \mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. It follows from Propositions 7 and 8 that the restriction of $\chi_{\overline{q}}$ to $\mathrm{N}(\Sigma(B_0))$ is homotopic to our original map f. Consequently, $\chi_{\overline{q}}$ determines a homotopy from f to a constant map; this proves (a).

Let us now prove (b). Suppose we are given an element $\eta \in \pi_n(|\mathcal{M}|, [X])$, represented by a map $f : B = \partial\,\Delta^{n+1} \to \mathcal{M}$. This map classifies a PL fibration $q : E \to |B|$, whose fiber over some fixed base point $b \in |B|$ coincides with X. Choosing compatible triangulations of E and $|B|$ (and arranging that our triangulation of $|B|$ contains b as a vertex) and applying Construction 5, we obtain a map $\chi_q : \mathrm{N}(\Sigma(|B|)) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. It is not hard to see that $\gamma(\chi_q)$ is homotopic to our original map f. This *almost* proves (b): it shows that every map from an n-sphere into $|\mathcal{M}|$ is *freely* homotopic to a map which factors through $|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})|$. However, this only tells us that η is conjugate under the action of $\pi_1(|\mathcal{M}|, [X])$ to an element which lies in the image of the map

$$\pi_n(|\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})|, [\Sigma(X)]) \to \pi_n(|\mathcal{M}|, [X]).$$

The problem is that the map χ_q need not be pointed: it carries the base point b to the partially ordered set $\Sigma(E_b)$ of simplices for the triangulation of E_b determined by our chosen triangulation of E, which might well differ from our original triangulation of X. However, Proposition 9 implies that there is a path from $[\Sigma(E_b)]$ to $[\Sigma(X)]$ whose image in $\mathcal{M}$ is homotopic to the identity, so that we can our free homotopy can be replaced by a based homotopy. $\square$

We now turn to the proofs of Propositions 7, 8, and 9. We will need a device for producing some homotopies.

Definition 10. Let $f : P \to Q$ be a Cartesian fibration of partially ordered sets. We will say that a pair $(p, p') \in P \times P$ is *Cartesian* if p is a largest element of $\{a \in P : a \leq p' \text{ and } f(a) \leq f(p)\}$.

Construction 11. Let $f : P \to P'$ be a map of partially ordered sets. We let $P' \amalg_f P$ denote the disjoint union of P and P', partially ordered so that there is a Cartesian fibration $g : P' \amalg_f P \to \{0 < 1\}$ with $P' = g^{-1}\{0\}$, $P = g^{-1}\{0\}$, and f the induced map from P to P'. In other words, the ordering on $P' \amalg_f P$ is defined so that $p' \leq p$ if and only if $p' \leq f(p)$ for $p \in P$ and $p' \in P'$.

Exercise 12. Suppose we are given a commutative diagram of partially ordered sets

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ \downarrow & & \downarrow \\ Q & \xrightarrow{g} & Q'. \end{array}$$

Show that the induced map $P' \amalg_f P \to Q' \amalg_g Q$ is a Cartesian fibration if and only if the following conditions are satisfied:

(i) The map f is a Cartesian fibration.

(ii) The map g is a Cartesian fibration.

(iii) The map f carries Cartesian pairs in P to Cartesian pairs in P'.

Exercise 13. In the situation of Exercise 12, show that the map $P' \amalg_f P \to Q' \amalg_g Q$ is good if and only if the following conditions are satisfied:

(i) The map f is good.

(ii) The map g is good.

(iii) For each $q \in Q$, the induced map $P_q \to P'_{g(q)}$ is left cofinal.

Remark 14. Let $u : P \to Q$ be a good Cartesian fibration of finite partially ordered sets. Then the construction $q \mapsto P_q$ determines a functor $Q \to \mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}$, hence a map of simplicial sets $\chi_u : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$.

Suppose we are given a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} \\ Q & \xrightarrow{g} & Q'. \end{array}$$

which satisfies the hypotheses described in Example 12 and 13. We note that $\mathrm{N}(Q')$ is a deformation retract of $\mathrm{N}(Q' \amalg_g Q)$. Moreover, the inclusion $\mathrm{N}(Q) \hookrightarrow \mathrm{N}(Q' \amalg_g Q)$ is a weak homotopy equivalence if and only if g is a weak homotopy equivalence. In this case, the Cartesian fibration $P' \amalg_f P \to Q' \amalg_g Q$ is classified by a map $\mathrm{N}(Q' \amalg_g Q) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ which determines a homotopy from χ_u to $\chi_{u'}$.

Proof of Proposition 8. For every partially ordered set P, let $T(P) = \{(p, \sigma) \in P \times \mathrm{Chain}(P) | p \leq \min(\sigma)\}$. Note that if $u : P \to Q$ is a Cartesian fibration, then the induced map $T(P) \to T(Q)$ is also a Cartesian fibration, where a pair $((p, \sigma), (p', \sigma'))$ is Cartesian if and only if the pairs (p, p') and (σ, σ') are individually Cartesian. Moreover, if u is good then the map $T(P) \to T(Q)$ is good. For each pair $(q, \sigma) \in T(Q)$, the projection maps

$$\mathrm{Chain}(P)_\sigma \leftarrow T(P)_{(q,\sigma)} \to P_q$$

are Cartesian fibrations with weakly contractible fibers, hence left cofinal. It follows that the diagrams

$$\begin{array}{ccccc} \mathrm{Chain}(P) & \longleftarrow & T(P) & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Chain}(Q) & \longleftarrow & T(Q) & \longrightarrow & Q \end{array}$$

satisfy the hypotheses of Remark 14, so that the maps $\chi_u : \mathrm{N}(Q) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ and $\chi_{\mathrm{Chain}(u)} : \mathrm{N}(\mathrm{Chain}(Q)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ are homotopic (since both are homotopic to $\chi_{T(u)}$). $\square$

Proof of Proposition 7. Let $q : E \to B$ be a PL fibration of polyhedra, and choose compatible triangulations of E and B; we will denote the simplices of these triangulations by $\Sigma(E)$ and $\Sigma(B)$, respectively. Suppose we are given PL homeomorphisms $E' \simeq E$ and $B' \simeq B$, where E' and B' are equipped with triangulations which refine the given triangulations on E and B (we can always arrange to be in this case, since any pair of (compatibly chosen) triangulations admits a common refinement). Set

$$P = \{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma' \subseteq \sigma\}$$

$$Q = \{(\tau', \tau) \in \Sigma(B') \times \Sigma(B) : \tau' \subseteq \tau\}.$$

The evident map $P \to Q$ is a Cartesian fibration. We claim that it is good: that is, for each $(\sigma'_0, \sigma_0) \in P$ and each $(\tau', \tau) \in Q$ containing $(q(\sigma'_0), q(\sigma_0))$, the partially ordered set

$$S = \{(\sigma', \sigma) \in P | q(\sigma') = \tau', q(\sigma) = \tau, \sigma'_0 \subseteq \sigma', \sigma_0 \subseteq \sigma\}$$

is weakly contractible. Note that since the map $\Sigma(E) \to \Sigma(B)$ is good, the partially ordered set $S' = \{\sigma \in \Sigma(E) | q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$ is weakly contractible. The projection map $S^{\mathrm{op}} \to S'^{\mathrm{op}}$ is Cartesian fibration, so it will suffice to show that it has weakly contractible fibers. The fiber over an element $\sigma \in S$ has the form

$$T = \{\sigma' \in \Sigma(E') | \sigma' \subseteq \sigma, q(\sigma') = \tau', \sigma'_0 \subseteq \sigma'\}.$$

This follows from the criterion of Lecture 9, since the projection map $\sigma \to \tau$ (a surjective map of simplices) is a fibration.

It is easy to see that the diagrams

$$\begin{array}{ccccc} \Sigma(E) & \longleftarrow & P & \longrightarrow & \Sigma(E') \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma(B) & \longleftarrow & Q & \longrightarrow & \Sigma(B') \end{array}$$

satisfy the criterion of Exercise 12. We claim that they also satisfy the criterion of Example 13. This amounts to two assertions:

- For every pair $(\tau', \tau) \in Q$, the natural map $P_{(\tau', \tau)} \to \Sigma(E)_\tau$ is left cofinal. For this, we must show that if $\sigma_0 \in \Sigma(E)$ satisfies $q(\sigma_0) = \tau$, then the partially ordered set

  $$S = \{(\sigma', \sigma) \in P : q(\sigma') = \tau', q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$$

  is weakly contractible. Let $S' = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$. Then S' is weakly contractible (it has a smallest element) and the map $S^{\mathrm{op}} \to S'^{\mathrm{op}}$ is a Cartesian fibration. It will therefore suffice to show that the fibers of $S \to S'$ are weakly contractible. That is, we must show that if $\sigma \in S'$, then the partially ordered set $\{\sigma' \in \Sigma(E') : \sigma' \subseteq \sigma, f(\sigma') = \tau'\}$ is weakly contractible. Again, this follows by applying our combinatorial criterion for fibrations to the map $\sigma \to \tau$.

- For every pair $(\tau', \tau) \in Q$, the natural map $P_{(\tau', \tau)} \to \Sigma(E')_{\tau'}$ is left cofinal. This map is a Cartesian fibration; it will therefore suffice to show that it has weakly contractible fibers. In other words, we must show that for $\sigma' \in \Sigma(E')$ with $q(\sigma') = \tau'$, the partially ordered set $S = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma' \subseteq \sigma\}$ is weakly contractible. Let σ_0 be the smallest simplex of E which contains σ', so we can write $S = \{\sigma \in \Sigma(E) : q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$. The weak contractibility of this partially ordered set follows from the fact that the map $E \to B$ is a fibration.

Applying Remark 14, we deduce that the good Cartesian fibrations

$$\Sigma(E) \to \Sigma(B) \qquad \Sigma(E') \to \Sigma(B')$$

determine homotopic maps $\mathrm{N}(\Sigma(B)) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$, $\mathrm{N}(\Sigma(B')) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$. $\square$

Proof of Proposition 9. We will show that the homotopy constructed in the proof of Proposition 7 has the desired property. Suppose we are given a PL homeomorphism of triangulated polyhedra $E' \to E$; as before, we may assume without loss of generality that the triangulation of E' refines the triangulation of E. Let $P = \{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma' \subseteq \sigma\}$ be defined as in the proof of Proposition 7 (in the special case $B = B' = *$). Let $\rho_1 : P \to \Sigma(E')$ and $\rho_2 : P \to \Sigma(E)$ be the projections onto the first and second factor, respectively. Then the partially ordered sets $Q = \Sigma(E) \amalg_{\rho_1} P$ and $Q' = \Sigma(E') \amalg_{\rho_2} P$ determine paths from $[\Sigma(E)]$ to $[P]$ and $[\Sigma(E')]$ to $[P]$ in $|\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})|$; we wish to prove that the images of these paths in $\mathcal{M}$ (which go from $[E]$ to $[|\mathrm{N}(P)|]$ are homotopic to one another. Let us denote these images by α and α', respectively.

We now construct some auxiliary partially ordered sets $\overline{Q}$ and $\overline{Q}'$ enlarging Q and Q', respectively. Let $\Sigma'(E)$ denote another copy of the partially ordered set $\Sigma(E)$ (given a different name to avoid confusion) and set

$$\overline{Q}' = Q' \amalg \Sigma'(X) = \Sigma(E') \amalg P \amalg \Sigma'(E),$$

equipped with the partial ordering where $q \leq q'$ if only if one of the following conditions holds:

- q and q' belong to $\Sigma(E')$ and $q \subseteq q'$.
- q and q' belong to P and $q \leq q'$ in P.
- q and q' belong to $\Sigma'(E)$ and $q \subseteq q'$.
- $q \in \Sigma(E')$, $q' = (\sigma', \sigma) \in P$, and $q \leq \sigma'$.
- $q \in \Sigma(E')$, $q' \in \Sigma'(X)$, and $q \subseteq q'$.
- $q = (\sigma', \sigma) \in P$, $q' \in \Sigma'(E)$, and $\sigma \subseteq q'$.

Let $\overline{Q} = Q \amalg \Sigma'(E) = \Sigma(E) \amalg P \amalg \Sigma'(E)$, which we regard as a partially ordered set in an analogous way. Note that we have evident maps of partially ordered sets

$$\overline{Q} \to \{0 < 1 < 2\} \leftarrow \overline{Q}'$$

Claim 15. *The PL maps*

$$|\mathrm{N}(\overline{Q})| \xrightarrow{u} \Delta^2 \xleftarrow{v} |\mathrm{N}(\overline{Q}')|$$

are fibrations.

For $i, j \in \{0, 1, 2\}$, let $\Delta^{\{i,j\}}$ denote the edge of Δ^2 containing i and j. It follows from Claim 15 that u and v are classified by maps $\overline{\alpha}, \overline{\alpha}' : \Delta^2 \to \mathcal{M}$ whose restrictions to $\Delta^{\{0,1\}}$ are the paths α and α', respectively. By construction, the inverse images $u^{-1}\Delta^{\{1,2\}}$ and $v^{-1}\Delta^{\{1,2\}}$ are canonically isomorphic and therefore determine the same path in $\mathcal{M}$. Consequently, to show that α and α' are homotopic, it suffices to show that the restrictions of $\overline{\alpha}$ and $\overline{\alpha}'$ to $\Delta^{\{0,2\}}$ are homotopic. Note that the inverse image $u^{-1}\Delta^{\{0,2\}}$ is canonically isomorphic to $E \times [0, 1]$, so that the restriction of $\overline{\alpha}$ to $\Delta^{\{0,2\}}$ is constant. To complete the proof, it suffices to show that the restriction of $\overline{\alpha}'$ to $\Delta^{\{0,2\}}$ is a nullhomotopic path from $[E']$ to $[E]$ (which we identify with each other). This can be established by showing that the the PL homeomorphism $E' \to E$ extends to a PL homoemorphism

$$v^{-1}\Delta^{\{0,2\}} = \mathrm{N}(\Sigma(E') \amalg \Sigma(E)) \to E \times [0, 1]$$

which restricts to the identity over the point $1 \in [0, 1]$ (the relevant homeomorphism can be constructed by defining it on vertices and extending linearly). □

Proof of Claim 15. We will prove that the map v is a fibration; the proof for u is similar (and slightly easier). Let $g : \overline{Q}' \to \{0, 1, 2\}$ be the projection map. For each nonempty subset $I \subseteq \{0, 1, 2\}$, we let $\mathrm{Chain}_I(\overline{Q}')$ denote the partially ordered set of chains in $\overline{Q}'$ whose image in $\{0, 1, 2\}$ is I. Recall that to show that v is a fibration, it will suffice to show that for $I \subseteq J$ the restriction map $\theta : \mathrm{Chain}_J(\overline{Q}') \to \mathrm{Chain}_I(\overline{Q}')$ has weakly contractible fibers. Moreover, we may assume without loss of generality that I is obtained from J by removing a single element. If $J = \{0, 1\}$, the desired result was established in Proposition 8. If $J = \{0, 2\}$ or $J = \{1, 2\}$, then the map $g^{-1}J \to J$ is the opposite of a Cartesian fibration; it will therefore suffice to show (by the opposite of our previous results) that the induced maps $\Sigma(E') \to \Sigma'(E)$ and $P \to \Sigma'(E)$ are right cofinal. The first statement is equivalent to the assertion that for $\sigma \in \Sigma'(E)$, the partially ordered set $\{\sigma' \in \Sigma(E') : \sigma' \subseteq \sigma\}$ is weakly contractible: this is clear, since the geometric realization of this poset is homeomorphic to σ. For the second statement, we observe that $P^{\mathrm{op}} \to \Sigma'(E)^{\mathrm{op}}$ is a Cartesian fibration; we are therefore reduced to proving that it has constructible fibers, and those fibers are again of the form $\{\sigma' \in \Sigma(E) : \sigma' \subseteq \sigma\}$.

It remains to treat the case where $J = \{0, 1, 2\}$ and $I = J - \{i\}$ for some $i \in \{0, 1, 2\}$. In the case $i = 0$ each fiber of θ has the form $\mathrm{Chain}(S)$ where S has a largest element, and in the case $i = 2$ each fiber of θ has the form $\mathrm{Chain}(S)$ where S has a smallest element. In the case $i = 1$, each fiber of θ has the form $\mathrm{Chain}(S)$ where S has the form

$$\{(\sigma', \sigma) \in \Sigma(E') \times \Sigma(E) : \sigma'_0 \subseteq \sigma' \subseteq \sigma \subseteq \sigma_0\}$$

for some fixed pair of simplices $\sigma'_0 \in \Sigma(E')$, $\sigma_0 \in \Sigma(E)$ with $\sigma'_0 \subseteq \sigma_0$. Let S' be the subset of S consisting of those pairs (σ', σ) where $\sigma' = \sigma_0$. The inclusion $S' \hookrightarrow S$ admits a right adjoint and is therefore a weak homotopy equivalence. It therefore suffices to show that S' is weakly contractible, which follows from the observation that S' has a largest element. □