Overview (Lecture 1)

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Let M and N be smooth closed manifolds of dimension n. An *h-cobordism* from M to N is a compact smooth manifold B of dimension $(n+1)$ with boundary $\partial B \simeq M \amalg N$ having the property that the inclusion maps $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences. If $n \geq 5$ and the manifold M is simply connected, then the celebrated *h-cobordism theorem* of Smale asserts that B is diffeomorphic to a product $M \times [0,1]$ (and, in particular, M is diffeomorphic to N). This theorem has many pleasant consequences.

Application 1 (Generalized Poincare Conjecture)**.** Let X be a smooth manifold of dimension d which has the homotopy type of a sphere S^d. Choose a pair of points $x, y \in X$ with $x \neq y$, and let X^0 be the manifold obtained from X by removing the interiors of nonoverlapping small disks around x and y. Then X^0 is a manifold with boundary $S^{d-1} \amalg S^{d-1}$. If $d \geq 3$, then removing the points x and y from X (or small disks around them) does not change the fundamental group of X, so that X^0 is simply connected. A simple calculation with homology shows that the inclusions of each boundary component into X^0 is a homotopy equivalence: that is, X^0 is an h-cobordism from S^{d-1} to itself. Applying the h-cobordism theorem, we deduce that M^0 is diffeomorphic to a product $S^{d-1} \times [0,1]$. The original manifold X can be recovered (up to homeomorphism) from X^0 by adjoining a cone on each of its boundary components. It follows that X is homeomorphic to the sphere S^d.

Remark 2. Beware that the manifold X need not be diffeomorphic to S^d: the identification of X^0 with $S^{d-1} \times [0,1]$ may not be compatible with the identification of ∂X^0 with $S^{d-1} \amalg S^{d-1}$.

In the non-simply connected case, it is generally not true that an h-cobordism from M to N is diffeomorphic to a product $M \times [0,1]$. To guarantee this, one needs a stronger hypothesis on the inclusion $M \hookrightarrow B$. Before introducing this hypothesis, we need to embark on a slight digression.

Definition 3. Let X be a finite simplicial complex. Suppose that there is a simplex $\sigma \subseteq X$ containing a face $\sigma_0 \subseteq \sigma$ such that σ is not contained in any larger simplex of X, and σ_0 is not contained in any larger simplex other than σ. Let $Y \subseteq X$ be the subcomplex obtained by removing the interiors of σ and σ_0. Then the inclusion $\iota : Y \hookrightarrow X$ is a homotopy equivalence. In this situation, we will say that ι is an *elementary expansion*. Note that Y is a retract of X; a retraction of X onto Y will be called an *elementary collapse*.

Definition 4. Let $f : Y \to X$ be a map between finite simplicial complexes. We will say that f is a *simple homotopy equivalence* if it is homotopic to a finite composition of elementary expansions and elementary collapses.

Any compact smooth manifold can be regarded as a finite simplicial complex (by choosing a triangulation), so it makes sense to talk about simple homotopy equivalence between smooth manifolds. One then has the following result:

Theorem 5 (s-Cobordism Theorem)**.** *Let B be an h-cobordism theorem between smooth manifolds M and N of dimension ≥ 5. Then B is diffeomorphic to a product $M \times [0,1]$ if and only if the inclusion map $M \hookrightarrow B$ is a simple homotopy equivalence.*

One of our main goals in this course is to formulate and prove a *parametrized* version of Theorem 5. Let us now outline the path we will take.

1 Theme 1: Higher Simple Homotopy Theory

The subject of "higher" simple homotopy theory begins with the following:

Question 6. Let B be a finite simplicial complex and suppose we are given a fibration $q : E \to B$. Under what conditions can we find another finite simplicial complex E' and a fibration $q' : E' \to B$ which is fiber-homotopy equivalent to q?

There is an obvious necessary condition: if $q : E \to B$ is fiber-homotopy equivalent to a fibration between finite simplicial complexes, then each fiber $\{E_b\}_{b \in B}$ must itself have the homotopy type of a finite complex. Let us say that a fibration $q : E \to B$ is *homotopy finite* if it satisfies this condition, and *finite* if the total space E is itself a finite complex.

For a fixed base B, there is a bijective correspondence between equivalence classes of homotopy finite fibrations $q : E \to B$ and homotopy classes of maps

$$B \to \amalg \, \mathrm{BAut}(X),$$

where the coproduct is taken over all homotopy equivalence classes of finite simplicial complexes X, and $\mathrm{Aut}(X)$ denotes the space of homotopy equivalences of X with itself. In other words, the coproduct $\amalg \, \mathrm{BAut}(X)$ is a *classifying space* for homotopy finite fibrations.

Following Hatcher ([2]), we will introduce another space $\mathcal{M}$ which enjoys the following analogous property: for any finite simplicial complex B, there is a bijective correspondence between homotopy classes of maps from B into $\mathcal{M}$ and equivalence classes of finite fibrations $q : E \to B$ where E is also a finite simplicial complex (here the right notion of equivalence is *concordance*; we will return to this point later). We can think of $\mathcal{M}$ as a sort of "moduli space of finite simplicial complexes." Every finite simplicial complex X will determine a point $[X] \in \mathcal{M}$, and one can show that two points $[X], [Y] \in \mathcal{M}$ belong to the same connected component of $\mathcal{M}$ if and only if there is a simple homotopy equivalence from X to Y.

2 Theme 2: Simple Homotopy Theory and Algebraic K-Theory

To apply the s-cobordism theorem in practice, one needs to address the following:

Question 7. Let $f : X \to Y$ be a homotopy equivalence of finite simplicial complexes (or manifolds). When is f is a simple homotopy equivalence?

This is a classical question which was answered by J.H.C. Whitehead. To each homotopy equivalence $f : X \to Y$ as above, one can associate an algebraic invariant $\tau(f)$ called the *Whitehead torsion* of f, which belongs to a certain abelian group $\mathrm{Wh}(X)$ called the *Whitehead group* of X. Whitehead proved that $\tau(f)$ vanishes if and only if f is a simple homotopy equivalence. Moreover, this is automatic in many cases: for example, if X is simply connected then the entire group $\mathrm{Wh}(X)$ is trivial (which is why one does not need to consider simple homotopy theory in the statement of the classical h-cobordism theorem).

One of our goals in this course will be to study "higher" analogues of the Whitehead torsion, which can be used to answer the following generalization of Question 7:

Question 8. What is the relationship between the spaces $\mathcal{M}$ and $\amalg \, \mathrm{BAut}(X)$? How close is the map θ to being a homotopy equivalence?

To address Question 8, Waldhausen introduced the subject that is now known as *Waldhausen K-theory*, or the *algebraic K-theory of spaces*. This theory associates to every space X a spectrum $A(X)$, called the *A-theory spectrum of* X. This construction is functorial in X. Consequently, to any space X one can associate an *assembly map*

$$A(*) \wedge X_+ \to A(X).$$

We will denote the cofiber of this map by $\mathbf{Wh}(X)$ and refer to it as the *(topological) Whitehead spectrum* of X. It is a generalization of the Whitehead group in the sense that there is a canonical isomorphism $\mathrm{Wh}(X) \simeq \pi_1 \mathbf{Wh}(X)$. One of the main theorems of this course will be the following result of Waldhausen:

Theorem 9. *Let Y be a finite simplicial complex. Then the homotopy fiber of the map $\theta : \mathcal{M} \to \amalg \mathrm{BAut}(X)$ taken at the point $[Y] \in \mathcal{M}$ is canonically homotopy equivalent to $\Omega^{\infty+1}\mathbf{Wh}(Y)$.*

Remark 10. Any homotopy equivalence of finite simplicial complexes $f : Y \to Z$ determines a path between the images of $[Y]$ and $[Z]$ in the $\amalg \mathrm{BAut}(X)$, and therefore allows us to lift $[Z]$ to a point η of the homotopy fiber

$$\mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[Y]\}.$$

Under the bijection $\pi_0(\mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[Y]\}) \simeq \pi_1 \mathbf{Wh}(Y)$ supplied by Theorem 9, the homotopy class of the point η corresponds to the Whitehead torsion $\tau(f)$. Consequently, the statement that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence can be regarded as a special case of Theorem 9.

Theorem 9 can be regarded as a definitive answer to Questions 8: it provides a purely "algebraic" description of the homotopy fibers of the map θ. Of course, to deduce any concrete consequences from this, one would need to understand the spectra $A(X)$: this is a very difficult problem about which much is known, but one which we will *not* consider in this course.

3 Theme 3: Simple Homotopy Theory as Stabilized Manifold Theory

There are several variants of Question 6 that one could consider:

Question 11. Let B be a finite simplicial complex and let $q : E \to B$ be a fibration. Under what conditions is q fiber-homotopy equivalent to a map $q' : E' \to B$ of finite simplicial complexes which exhibits E' as a *fiber bundle* over B?

At first glance, it would seem that Question 11 should have a very different answer than Question 8. Suppose that x and y are two points which belong to the same path component of B. If $q : E \to B$ is a fibration, then we can conclude that the fibers E_x and E_y are homotopy equivalent. If $q : E \to B$ is a finite fibration, then higher simple homotopy theory will tell us that E_x and E_y are related by a *simple* homotopy equivalence. But if $q : E \to B$ is a fiber bundle, then it follows E_x and E_y are homeomorphic: a dramatically stronger conclusion.

Somewhat surprisingly, it turns out that the answers to Questions 6 and 11 are the same:

Theorem 12. *Let $q : E \to B$ be a fibration between finite simplicial complexes. Then q is fiber-homotopy equivalent to a fiber bundle $q' : E' \to B$. Moreover, one can arrange that the fibers of q' are (piecewise-linear) manifolds (with boundary).*

In fact, one can be more precise. For each integer $d \geq 0$, one can define a classifying space $\mathcal{M}_d$ for framed (piecewise-linear) manifolds with boundary. There are stabilization maps

$$\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots$$

given by forming the product with the interval $[0,1]$. Another of our goals in this course will be to prove the following result of Chapman:

Theorem 13. *The direct limit $\varinjlim \mathcal{M}_d$ is homotopy equivalent to $\mathcal{M}$.*

Remark 14. Let K be a finite simplicial complex. Then K is always homotopy equivalent to a framed (piecewise-linear) manifold M (with boundary). To find M, one can choose a piecewise-linear embedding of K into some Euclidean space $\mathbb{R}^n$ and take a regular neighborhood of K (with respect to a sufficiently fine triangulation of $\mathbb{R}^n$). Theorem 13 says roughly that it is possible to perform this construction "in families."

4 Theme 4: The Classification of Manifolds

Fix an integer $d \geq 0$. For each closed d-manifold M, let $\mathrm{Homeo}(M)$ denote the group of homeomorphisms of M with itself. We will refer to the disjoint union (taken over all homeomorphism classes of closed d-manifolds)

$$\mathcal{N} = \coprod_M \mathrm{BHomeo}(M)$$

as the *classifying space for manifolds of dimension d*. For any nice space B, there is a bijection between homotopy classes of maps from B into $\mathcal{N}$ and fiber bundles over B whose fibers are compact d-manifolds.

Remark 15. In this overview, we will focus our attention on topological manifolds and homeomorphisms, rather than smooth manifolds and diffeomorphisms. All results have analogues in the smooth setting, but many of the statements are a bit more complicated.

One basic problem in high-dimensional topology is to understand the homotopy type of the classifying space $\mathcal{N}$ (equivalently, to understand the homotopy type of the homeomorphism groups $\mathrm{Homeo}(M)$). As a first step, one can approximate $\mathcal{N}$ by another space $\mathcal{N}^h$, which classifies manifolds "up to h-cobordism" rather than up to homeomorphism. This approximation is much easier to understand: for $d \geq 5$, there is a completely algebraic description of $\mathcal{N}^h$ in terms of algebraic L-theory. Consequently, the problem of understanding the classifying space $\mathcal{N}$ is reduced to the problem of understanding the homotopy fibers of the map $\mathcal{N} \to \mathcal{N}^h$. This requires us to analyze the problem of "straightening out" h-cobordisms:

Question 16. Let M be a compact manifold and let $q : E \to B$ be a fiber bundle, where each fiber E_b is an h-cobordism from M to some other manifold N_b. When is E isomorphic to a product $M \times [0,1] \times B$?

When the base B is a point, the s-cobordism theorem gives a complete answer to Question 16 (at least for $\dim(M) \geq 5$): the h-cobordism E is a product if and only if the inclusion $f : M \hookrightarrow E$ is a simple homotopy equivalence, or equivalently if the Whitehead torsion $\tau(f)$ vanishes. Another of our main objectives in this course is to study a *parametrized* s-cobordism theorem:

Theorem 17 (Parametrized Stable s-Cobordism Theorem)**.** *Let $q : E \to B$ be as in Question 16, where B is a finite simplicial complex. Then there is an obstruction $\tau(E) \in \mathbf{Wh}(M)^{-1}(B)$ which vanishes if and only if there is a fiberwise equivalence*

$$E \times [0,1]^k \simeq M \times [0,1]^{k+1} \times B$$

for $k \gg 0$.

Remark 18. The obstruction $\tau(E)$ is given by the homotopy class of the map

$$B \to \Omega^{\infty+1}\mathbf{Wh}(M) \simeq \mathcal{M} \times_{\amalg \mathrm{BAut}(X)} \{[M]\}$$

classifying the finite fibration $E \to B$ and its fiber-homotopy equivalence to the constant bundle $M \times B \to B$).

Warning 19. As stated, Theorem 17 does not imply the (topological) s-cobordism theorem. In the special case where B is a point, it asserts that an h-cobordism E with vanishing Whitehead torsion becomes homeomorphic to $M \times [0,1]$ after forming a product with $[0,1]^k$ for $k \gg 0$. The s-cobordism theorem says that we can take $k = 0$ provided that $\dim(M) \geq 5$.

One might hope for a common generalization which asserts that if $\tau(E) = 0$ and $\dim(M)$ is sufficiently large (compared with $\dim(B)$), then E is equivalent to $M \times [0,1] \times B$.

5 Theme 5: Infinite-Dimensional Topology

Theorems 12 and 17 suggest that higher simple homotopy theory can be viewed as a "stabilized" theory of manifolds where we allow the dimension to grow. It turns out that many aspects of the theory can be elucidated by considering manifolds of infinite dimension, such as the Hilbert cube

$$Q = [0,1] \times [0,1] \times \cdots .$$

For example, one has the following theorem of Chapman:

Theorem 20. *Let $f : X \to Y$ be a map of finite simplicial complexes. Then f is a simple homotopy equivalence if and only if the induced map $X \times Q \to Y \times Q$ is homotopic to a homeomorphism.*

Theorem 20 greatly expands the usefulness of simple homotopy theory. For example, it implies that the Whitehead torsion is a *topological* invariant, which is not *a priori* clear from its definition (and was not known before Theorem 20).

If time permits, we'll discuss the proof of Theorem 20 and some related results (many of which are also due to Chapman) which imply that the moduli space $\mathcal{M}$ admits several "infinite-dimensional" descriptions:

- It is a classifying space for fibrations whose fibers are compact ANRs.
- It is classifying space for fiber bundles whose fibers are compact and locally homeomorphic to Q (such spaces are called *Hilbert cube manifolds*).