Math 261y: von Neumann Algebras (Lecture 18)

October 17, 2011

Let A be an abelian von Neumann algebra. Our goal in this lecture is to study the Hilbert space representations of A. Let us write $A = L^\infty(X)$ for some measure space X. We have seen that a representation $\rho : A \rightarrow B(V)$ is determined by its restriction to the projections in A. We can identify the collection of projections with Σ/Σ_0, where Σ is the collection of measurable subsets of X and Σ_0 is the collection of sets of measure zero. To every such projection, ρ associates a projection operator on V, which we can identify with a closed subspace of V. This determines a construction

$$\mu : \{ \text{ measurable subsets of X } \} \rightarrow \{ \text{ closed subspaces of } V \}$$

satisfying the following condition:

(a) If $\{Y_\alpha\}$ is a collection of measurable subsets of X which are disjoint modulo sets of measure zero, then the subspaces $\mu(Y_\alpha)$ are mutually orthogonal and

$$\mu(\bigvee Y_\alpha) = \bigoplus \mu(Y_\alpha)$$

where $\bigvee Y_\alpha$ denotes a supremum for $\{Y_\alpha\}$ in the complete Boolean algebra Σ/Σ_0 (note that, if the collection $\{Y_\alpha\}$ is countable, we can just take $\bigvee Y_\alpha = \bigcup Y_\alpha$).

(b) We have $\mu(X - Y) = \mu(Y)^\perp$ for every measurable subset $Y \subseteq X$.

A map μ satisfying (a) and (b) is called a *spectral measure* on the measure space X. The recipe above determines a bijection between (ultraweakly continuous) representations of $L^\infty(X)$ on V and spectral measures on X (taking values in closed subspaces of V.

If we are willing to introduce some countability assumptions, we can say much more about the representation of A. Let us assume that A is separable and write $A = L^\infty(X)$ where (X, Σ, μ) is a standard finite measure space. For convenience we will make the following assumption:

- The σ-algebra Σ is complete (that is, if $K \subseteq X$ is a set of measure zero, then every subset of K is measurable). Note that this is departure from our earlier conventions.

Definition 1. A *field of Hilbert spaces* on X consists of the following data: for each point $x \in X$, a Hilbert space V_x, whose inner product we will denote by $(,)_x$. A *measurable field of Hilbert spaces* on X is a field of Hilbert spaces together with a subset

$$V_{\text{meas}} \subseteq \prod_{x \in X} V_x,$$

which we call the set of *measurable sections* of $\{V_x\}_{x \in X}$ such that there exists a countable collection $f_1, f_2, \ldots \in V_{\text{meas}}$ satisfying the following conditions:

(1) For each $x \in X$, the vectors $f_i(x)$ span a dense subset of the Hilbert space V_x.

(2) An arbitrary section $g \in \prod_{x \in X} V_x$ belongs to V_{meas} if and only if, for each $i \geq 1$, the function $x \mapsto (g(x), f_i(x))_x$ is measurable.

Remark 2. Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X. The set V_{meas} of measurable sections has the following properties:

- If $f, g \in V_{\mathrm{meas}}$, then $f + g \in V_{\mathrm{meas}}$.
- If $f \in V_{\mathrm{meas}}$ and $\lambda : X \to \mathbf{C}$ is a measurable function, then $\lambda f \in V_{\mathrm{meas}}$.
- If g_i is a sequence of elements of V_{meas} which converge pointwise almost everywhere to another element $g \in \prod_{x \in X} V_x$, then $g \in V_{\mathrm{meas}}$.

Remark 3. Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X. We will say that a sequence $f_1, f_2, \ldots \in V_{\mathrm{meas}}$ is a *generating sequence* if it satisfies conditions (1) and (2) of Definition 1.

Given an arbitrary sequence $f_1, f_2, \ldots \in V_{\mathrm{meas}}$, we will say that the sequence $\{f_i\}$ is *normalized* if, for each $x \in X$, the vectors $\{f_i(x) \in V_x\}_{i \geq 1}$ are mutually orthogonal and satisfy $||f_i(x)|| \in \{0, 1\}$.

Suppose that $f_1, f_2, \ldots$ is a generating sequence. Rescaling, we can assume that $f_1(x)$ is either zero or a unit vector for each $x \in X$. The construction $x \mapsto (f_2(x), f_1(x))_x$ determines a measurable function $\lambda : X \to \mathbf{C}$. Replacing f_2 by $f_2 - \lambda f_1$, we can reduce to the case where $f_2(x) \perp f_1(x)$ for all x. Rescaling, we may further assume that $f_2(x)$ is either zero or a unit vector for every x. Proceeding in this way, we can replace any generating sequence by a normalized sequence; it is easy to see that this is also a generating sequence.

Suppose that $f_1, f_2, \ldots \in V_{\mathrm{meas}}$ is a normalized generating sequence. For any $g \in \prod_{x \in X} V_x$, if we set $\lambda_i(x) = (g(x), f_i(x))_x$, then the sum $\sum_{i \geq 1} \lambda_i f_i$ converges pointwise to g. Note that g is measurable if and only if each $\lambda_i : X \to \mathbf{C}$ is a measurable function. If $g, g' \in V_{\mathrm{meas}}$ and we set $\lambda'_i(x) = (g'(x), f_i(x))_x$, then the sum $\sum_{i \geq 1} \lambda'_i f_i$ converges pointwise to g'. It follows that the sum

$$\sum_{i,j \geq 1} \lambda_i \overline{\lambda}'_i$$

converges pointwise to the function $x \mapsto (g(x), g'(x))_x$. In particular, $(g(x), g'(x))_x$ is a measurable function.

Lemma 4. *Let $(\{V_x\}_{x \in X}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X. Suppose that $f_1, f_2, \ldots \in V_{\mathrm{meas}}$ is a sequence of sections satisfying condition* (2) *of Definition 1. Then the sequence f_i is generating: that is, f_i also satisfies condition* (1).

Proof. Applying the Gram-Schmidt procedure to the f_i, we can assume that the sequence f_i is normalized. Condition (2) implies that the $f_i(x)$ form an orthonormal basis (possibly with some zero vectors inserted) for V_x, for each $x \in X$. It follows that for each $g \in \prod_{x \in X} V_x$, if we set $\lambda_i(x) = (g(x), f_i(x))_x$, then the sum

$$\sum_{i \geq 1} \lambda_i f_i$$

converges pointwise to g. If each λ_i is measurable, it follows that g is measurable. □

Definition 5. Let $(\{V_x\}_{x \in X}, V_{\mathrm{meas}})$ and $(\{V'_x\}_{x \in X}, V'_{\mathrm{meas}})$ be measurable fields of Hilbert spaces on X. A *bounded map of measurable fields* from $(\{V_x\}_{x \in X}, V_{\mathrm{meas}})$ to $(\{V'_x\}_{x \in X}, V'_{\mathrm{meas}})$ is a collection of bounded operators $F_x : V_x \to V'_x$ with the following properties:

(*i*) The induced map $\prod_{x \in X} V_x \to \prod_{x \in X} V'_x$ carries V_{meas} into V'_{meas}.

(*ii*) The map $x \mapsto ||F_x||$ is essentially bounded.

Given two bounded maps of fields $F_x, G_x : V_x \to V'_x$, we say that $\{F_x\}$ and $\{G_x\}$ are *equivalent* if $F_x = G_x$ for almost every $x \in X$.

We let $\mathcal{C}_X$ denote the category whose objects are measurable fields of Hilbert spaces, and whose morphisms are equivalence classes of bounded maps of fields.

Remark 6. To verify condition (i) of Definition 5, it suffices to check that the induced map $\prod_{x\in X} V_x \to \prod_{x\in X} V'_x$ carries a normalized generating sequence for $\{V_x\}$ into V'_{meas}.

Construction 7. Let $(\{V_x\}_{x\in X}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X. We say that two elements $f, g \in V_{\text{meas}}$ are *equivalent* if $f(x) = g(x)$ for almost every x. We say that an element $f \in V_{\text{meas}}$ is *square-integrable* if the function $x \mapsto (f(x), f(x))_x$ is an integrable function on X. We let $V^{(2)}_{\text{meas}}$ denote the set of equivalence classes of square integrable elements of V_{meas}. We equip $V^{(2)}_{\text{meas}}$ with an inner product given by

$$(f, g) = \int_X (f(x), g(x))_x d\mu.$$

Lemma 8. *Let $(\{V_x\}_{x\in X}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X. Then $V^{(2)}_{\text{meas}}$ is a separable Hilbert space, acted on by the von Neumann algebra $L^\infty(X)$.*

Proof. For $f, g \in V^{(2)}_{\text{meas}}$, the well-definedness of the inner product (f, g) follows from the inequalities

$$\int_X |(f(x), g(x))_x| d\mu \leq \int_X ||f(x)||||g(x)|| d\mu \leq (\int_X ||f(x)||^2 d\mu)^{\frac{1}{2}} (\int_X ||g(x)||^2 d\mu)^{\frac{1}{2}} < \infty.$$

It is easy to see that $(,)$ defines a norm on $V^{(2)}_{\text{meas}}$. Choose a normalized generating sequence $\{f_i \in V_{\text{meas}}\}$. For each i, set $X_i = \{x \in X : f_i(x) \neq 0\}$. Then the construction $(\lambda_i)_{i\geq 1} \mapsto \sum \lambda_i f_i$ induces an isometry

$$\bigoplus_{i\geq 1} L^\infty(X_i) \to V^{(2)}_{\text{meas}},$$

from which we deduce that $V^{(2)}_{\text{meas}}$ is a separable Hilbert space.

It is easy to see that if $\lambda : X \to \mathbf{C}$ is essentially bounded and $f \in V_{\text{meas}}$ is square-integrable, then λf is square integrable: thus $L^\infty(X)$ acts on $V^{(2)}_{\text{meas}}$ by multiplication. In fact, this action is ultraweakly continuous. To prove this, it suffices to show that for every $f \in V_{\text{meas}}$ and every collection of pairwise disjoint measurable sets $\{E_\alpha\}_{\alpha\in I}$ with characteristic functions χ_{E_α}, the collection of partial sums $\sum_{\alpha\in I_0} \chi_{E_\alpha} f$ converges to $\chi_E f$ in $V^{(2)}_{\text{meas}}$, where χ_E is the sum of the projections χ_{E_α} in $L^\infty(X)$. Since X is standard, the set I can be taken to be countable, so we can just write $E = \bigcup E_\alpha$. Unwinding the definitions, we wish to show that

$$\int ||f(x)||^2 (\chi_E - \bigcup_{\alpha\in I_0} \chi_{E_\alpha}) d\mu$$

converges to zero, which follows from the dominated convergence theorem. □

We are now ready to state the main result of this lecture:

Theorem 9. *Let X be a standard probability space. The construction*

$$\Psi : (\{V_x\}_{x\in X}, V_{\text{meas}}) \mapsto V^{(2)}_{\text{meas}}$$

determines an equivalence from the category $\mathcal{C}_X$ of measurable fields of Hilbert spaces on X to the category of separable Hilbert space representations of $L^\infty(X)$.

Proof. We first claim that the functor Ψ is faithful. That is, if we are given a bounded map of fields $F_x : V_x \to V'_x$ which induces the zero map $V^{(2)}_m \to V'^{(2)}_m$, then $F_x = 0$ for almost every x. To prove this, choose a normalized generating sequence f_i, so that each f_i belongs to $V^{(2)}_{\text{meas}}$. If $\{F_x\}$ annihilates f_i, then we deduce that $F_x(f_i(x)) = 0$ for almost every x. Since there are only countably many f_i's, we deduce that $F_x(V_x) = 0$ for almost every x.

We next prove that the functor Ψ is full. Suppose we are given a bounded operator measurable fields $(\{V_x\}, V_{\text{meas}})$ and $(\{V'_x\}, V'_{\text{meas}})$, and a bounded operator $F : V^{(2)}_{\text{meas}} \rightarrow V'^{(2)}_{\text{meas}}$ which commutes with the action of $L^\infty(X)$. Let $C = ||F||$. Choose normalized generating sequences f_i and f'_j. We have almost everywhere defined measurable functions

$$\lambda_{i,j} : X \rightarrow \mathbf{C}$$

given by $\lambda_{i,j}(x) = (F(f_i)(x), f'_j(x))_x$. We claim that the construction

$$v \mapsto \sum_{i,j} (v, f_i(x))_x \lambda_{i,j}(x) f'_j(x)$$

determines a bounded operator $F_x : V_x \rightarrow V'_x$ with $||F_x|| \leq C$ for almost every x. Assume this for the moment. Since $F_x(f_i(x)) = F(f_i)(x)$ for almost every x, we conclude that $\{F_x\}$ carries a generating sequence for $\{V_x\}$ into V'_{meas}, so that $\{F_x\}$ is a bounded map of measurable fields (Remark 6). Let $G : V^{(2)}_m \rightarrow V'^{(2)}_m$ be the bounded operator determined by the family $\{F_x\}$. By construction, we have

$$G(f_i) = F(f_i).$$

It follows that F and G coincide on all finite $L^\infty(X)$-linear combinations of the sections f_i. Since these combinations are dense in $V^{(2)}_{\text{meas}}$, we conclude that $F = G$.

We will complete the proof of this result in the next lecture. □