The Adams Spectral Sequence (Lecture 8)

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Recall that our goal this week is to prove the following result:

Theorem 1 (Quillen). *The universal complex orientation of the complex bordism spectrum* MU *determines a formal group law over* $\pi_* \mathrm{MU}$. *This formal group law is classified by an isomorphism of commutative rings* $L \to \pi_* \mathrm{MU}$.

To prove this theorem, we need a method for calculating the homotopy groups $\pi_* \mathrm{MU}$. In the last lecture, we computed the homology groups $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$. The universal coefficient theorem then gives $\mathrm{H}_*(\mathrm{MU}; R) \simeq R[b_1, b_2, \ldots]$ for any commutative ring R. In this lecture, we review a general method for passing from information about the homology of a spectrum to information about its homotopy groups: the *Adams spectral sequence*.

Fix a prime number p, and let R denote the Eilenberg-MacLane spectrum $H\mathbf{F}_p$. Then E admits a coherently associative multiplication. In particular, we can define a functor $\overline{R}^{\bullet}$ from finite linearly ordered sets to spectra, given by $\{0, 1, \ldots, n\} \mapsto R \otimes \cdots \otimes R \simeq R^{\otimes n+1}$. In other words, we can view $\overline{R}^{\bullet}$ as an *augmented cosimplicial spectrum*. Restricting to *nonempty* finite linearly ordered sets, we get a cosimplicial spectrum which we will denote by $R^{\bullet}$. If X is any other spectrum, we can define an augmented cosimplicial spectrum $\overline{X}^{\bullet} = X \otimes \overline{R}^{\bullet}$; let $X^{\bullet}$ denote the underlying cosimplicial spectrum. The augmented cosimplicial spectrum $\overline{X}^{\bullet}$ determines a map

$$X \simeq X \otimes S \simeq X \otimes \overline{R}^{-1} = \overline{X}^{-1} \to \mathrm{Tot}\, X^{\bullet}.$$

Put more concretely, we have the canonical Adams resolution of X, which is a chain complex of spectra

$$X \to X \otimes R \overset{d^0 - d^1}{\to} X \otimes R \otimes R \overset{d^0 - d^1 + d^2}{\to} X \otimes R \otimes R \otimes R \to \cdots$$

Any cosimplicial spectrum $X^{\bullet}$ determines a spectral sequence $\{E_r^{a,b}\}$. Here $E_1^{a,b} = \pi_a X^b$, and the differential on the first page is the differential in the chain complex

$$\pi_a X^0 \overset{d^0 - d^1}{\to} \pi_a X^1 \overset{d^0 - d^1 + d^2}{\to} \pi_a X^2 \to \cdots$$

In good cases, this spectral sequence will converge to information about the totalization $\mathrm{Tot}\, X^{\bullet}$. In our case, we have the following result:

Theorem 2 (Adams). *Let* X *be a connective spectrum whose homotopy groups* $\pi_n X$ *are finitely generated for every integer* n. *Fix a prime number* p *and let* $R = H\mathbf{F}_p$, *and let* $X \to \mathrm{Tot}\, X^{\bullet}$ *be the map constructed above. Then:*

(1) *For every integer* n, *we have a canonical decreasing filtration*

$$\cdots \subseteq F^2 \pi_n X \to F^1 \pi_n X \to F^0 \pi_n X = \pi_n X$$

where $F^i \pi_n X$ *is the kernel of the map* $\pi_n X \to \pi_n \mathrm{Tot}^{i-1} X^{\bullet}$.

(2) *The decreasing filtration $F^i\pi_n X$ is commensurate with the p-adic filtration. That is, for each $i \geq 0$, there exists $j \gg i$ such that $p^i\pi_n X \subseteq F^j\pi_n X \subseteq p^j\pi_n X$. In particular, we have a canonical isomorphism*

$$\varprojlim(\pi_n X/F^j\pi_n X) \simeq \varprojlim(\pi_n X/p^i\pi_n X) \simeq (\pi_n X)^\vee,$$

where $\vee$ denotes the functor of p-adic completion.

(3) *For fixed a and b, the abelian groups $\{E_r^{a,b}\}_{r\geq 0}$ stabilize to some fixed value $E_\infty^{a,b}$ for $r \gg 0$. Moreover, we have a canonical isomorphism*

$$F^b\pi_{a-b}(X)/F^{b+1}\pi_{a-b}X \simeq E_\infty^{a,b}.$$

If X is a ring spectrum, then $X^\bullet$ has the structure of a cosimplicial ring spectrum. In this case, we have the following additional conclusions:

(4) *For integers m and n, the multiplication map $\pi_m X \times \pi_n X \to \pi_{n+m}X$ carries $F^i\pi_m X \times F^j\pi_n X$ into $F^{i+j}\pi_{m+n}X$. In particular, we get a bilinear multiplication $E_\infty^{a,b} \times E_\infty^{a',b'} \to E_\infty^{a+a',b+b'}$.*

(5) *The spectral sequence $\{E_r^{a,b}\}$ is multiplicative. That is, for each r, we have bilinear maps $E_r^{a,b}\times E_r^{a',b'} \to E_r^{a+a',b+b'}$. These maps are associative in the obvious sense and compatible with the differential (i.e., the differential satisfies the Leibniz rule). Moreover, when $r \gg 0$ so that $E_r^{a,b} = E_\infty^{a,b}$ and $E_r^{a',b'} = E_\infty^{a',b'}$, these multiplications agree with the multiplications defined in* (4)*.*

To apply Theorem 2 in practice, we would like to understand the initial pages of the spectral sequence $\{E_r^{a,b}\}$. When $r = 1$, we have $E_r^{a,b} \simeq \pi_a(X\otimes R^{\otimes b+1})$. In particular, $E_1^{*,0} \simeq \pi_*(X\otimes R) = \mathrm{H}_*(X; \mathbf{F}_p)$ is the mod p homology of the spectrum X. To understand the next term, we write $X^1 = X\otimes R\otimes R = (X\otimes R)\otimes_R(R\otimes R)$. This gives a canonical isomorphism $E_1^{*,1} \simeq \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \pi_*(R \otimes R)$.

Definition 3. Let R be the ring spectrum $H\mathbf{F}_p$. The graded-commutative ring $\pi_*(R\otimes R)$ is called the *dual Steenrod algebra*, and will be denoted by $\mathcal{A}^\vee$.

More generally, we can write

$$X^b = X \otimes R^{\otimes b+1} = (X \otimes R) \otimes_R (R \otimes R) \otimes_R \cdots \otimes_R (R \otimes R).$$

This identification gives a canonical isomorphism

$$E_1^{*,b} \simeq \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} (\mathcal{A}^\vee)^{\otimes b}$$

. We have a chain complex of graded abelian groups

$$\mathrm{H}_*(X; \mathbf{F}_p) \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \to \cdots$$

associated to a cosimplicial graded abelian group $\mathrm{H}_*(X; \mathbf{F}_p) \otimes (\mathcal{A}^\vee)^{\otimes\bullet}$.

To describe the second page of the spectral sequence $\{E_r^{a,b}\}$, we would like understand the differentials in this chain complex. We begin by noting that $\mathcal{A}^\vee$ is actually a Hopf algebra. That is, there is a comultiplication $c : \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes_{\mathbf{F}_p} \mathcal{A}^\vee$, which is induced by the map of ring spectra

$$R \otimes R \simeq R \otimes S \otimes R \to R \otimes R \otimes R \simeq (R \otimes R) \otimes_R (R \otimes R)$$

by passing to homotopy groups. Moreover, this coalgebra *acts* on $\mathrm{H}_*(X; \mathbf{F}_p)$ for any spectrum X: that is, we have a canonical map $a : \mathrm{H}_*(X; \mathbf{F}_p) \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee$. This map is induced by the map of spectra

$$X \otimes R \simeq X \otimes S \otimes R \to X \otimes R \otimes R \simeq (X \otimes R) \otimes_R (R \otimes R).$$

Remark 4. Passing to (graded) vector space duals, we see that the dual of $\mathcal{A}^\vee$ is an algebra (called the *Steenrod algebra*, which acts on the cohomology $\mathrm{H}^*(X; \mathbf{F}_p)$ of any spectrum.

Unwinding the definitions, we see that each of the differentials

$$\mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}(\mathcal{A}^\vee)^{\otimes n-1}\to \mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}(\mathcal{A}^\vee)^{\otimes n}$$

is given by an alternating sum $\sum_{0\leq i\leq n} d^i$, where:

- The map d^0 is induced by the action map $a:\mathrm{H}_*(X;\mathbf{F}_p)\to \mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}\mathcal{A}^\vee$.
- The maps $d^1,\ldots,d^{n-1}$ are induced by the comultiplication $c:\mathcal{A}^\vee\to\mathcal{A}^\vee\otimes_{\mathbf{F}_p}\mathcal{A}^\vee$, applied to each factor of $\mathcal{A}^\vee$.
- The map d^n is given by the inclusion of the unit $\mathbf{F}_p\to\mathcal{A}^\vee$.

It is convenient to describe the above analysis in the language of algebraic geometry. For simplicity, we will henceforth assume that $p=2$, so that graded-commutative rings are actually commutative.

Proposition 5. (1) *Let $\mathbb{G}$ denote the spectrum of the commutative ring $\mathcal{A}^\vee$. The comultiplication $\mathcal{A}^\vee\to\mathcal{A}^\vee\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ determines a multiplication $\mathbb{G}\times_{\operatorname{Spec}\mathbf{F}_2}\mathbb{G}\to\mathbb{G}$, which endows $\mathbb{G}$ with the structure of an affine group scheme over* $\operatorname{Spec}\mathbf{F}_2$.

(2) *For any spectrum X, the action map $\mathrm{H}_*(X;\mathbf{F}_2)\to\mathrm{H}_*(X;\mathbf{F}_2)\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ endows the vector space $V=\mathrm{H}_*(X;\mathbf{F}_2)$ with the structure of a representation of the group scheme $\mathbb{G}$.*

(3) *The E_1-term of the Adams spectral sequence can be identified with the canonical cochain complex*

$$V\to\Gamma(\mathbb{G};V\otimes\mathcal{O}_{\mathbb{G}})\to\Gamma(\mathbb{G}^2;V\otimes\mathcal{O}_{\mathbb{G}^2})\to\cdots$$

which encodes the action of $\mathbb{G}$ on V.

(4) *The E_2-term of the Adams spectral sequence can be identified with the cohomologies of this cochain complex. In other words, we have*

$$E_2^{*,b}\simeq\mathrm{H}^b(\mathbb{G};\mathrm{H}_*(X;\mathbf{F}_2)).$$

In the special case where X is a ring spectrum, we can say more. In this case, the multiplication on X endows the homology $\mathrm{H}_*(X;\mathbf{F}_2)$ with the structure of a commutative $\mathbf{F}_2$-algebra. Then the spectrum $\operatorname{Spec}\mathrm{H}_*(X;\mathbf{F}_2)$ is an affine scheme Y. The action $\mathrm{H}_*(X;\mathbf{F}_2)\to\mathrm{H}_*(X;\mathbf{F}_2)\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ is a map of commutative rings, which determines a map of affine schemes $\mathbb{G}\times_{\operatorname{Spec}\mathbf{F}_2}Y\to Y$. In other words, the affine scheme Y is *acted on* by the group scheme $\mathbb{G}$. Moreover, the cohomology groups $\mathrm{H}^b(\mathbb{G};\mathrm{H}_*(X;\mathbf{F}_2))$ are simply the cohomology groups of the quotient stack $Y/\mathbb{G}$. In particular, we get an isomorphism of commutative algebras $E_2^{*,*}\simeq\mathrm{H}^*(Y/\mathbb{G};\mathcal{O}_{Y/\mathbb{G}})$.

To apply this information in practice, we need to understand the algebraic group $\mathbb{G}$. For each integer n, let $X(n)$ denote the function spectrum $S^{\mathbf{R}P^n}$, where $\mathbf{R}P^n$ denotes real projective space of dimension n. Then $X(n)$ is a commutative ring spectrum (in fact, an E_∞-ring spectrum), and we have a canonical isomorphism

$$\mathrm{H}_*(X(n);\mathbf{F}_2)\simeq\mathrm{H}^*(\mathbf{R}P^n;\mathbf{F}_2)\simeq\mathbf{F}_2[x]/(x^{n+1}).$$

In particular, we get an action of $\mathbb{G}$ on the affine scheme $\operatorname{Spec}\mathbf{F}_2[x]/(x^{n+1})$. Passing to the limit as n grows, we get an action of $\mathbb{G}$ on the formal scheme

$$\varinjlim\operatorname{Spec}\mathbf{F}_2[x]/(x^{n+1})\simeq\operatorname{Spf}\mathbf{F}_2[[x]]=\operatorname{Spf}H^*(\mathbf{R}P^\infty,\mathbf{F}_2)$$

This action is not arbitrary. Note that $\mathbf{R}P^\infty$ has a commutative multiplication. For example, we can realize $\mathbf{R}P^\infty$ as the projectivization of the *real* vector space $\mathbb{R}[t]$, and $\mathbf{R}P^n$ as the projectivization of the subspace of $\mathbb{R}[t]$ spanned by polynomials of degree $\leq n$. The multiplication on $\mathbb{R}[t]$ induces a multiplication $\mathbf{R}P^\infty\times\mathbf{R}P^\infty\to\mathbf{R}P^\infty$, which is the direct limit of multiplication maps $\mathbf{R}P^m\times\mathbf{R}P^n\to\mathbf{R}P^{m+n}$. Each of these multiplication maps induces a map of spectra $X(m+n) \to X(m) \otimes X(n)$, which induces a $\mathbb{G}$-equivariant map

$$\operatorname{Spec}(\mathbf{F}_2[x]/(x^{m+1})) \times_{\operatorname{Spec} \mathbf{F}_2} \operatorname{Spec}(\mathbf{F}_2[x]/(x^{n+1})) \to \operatorname{Spec}(\mathbf{F}_2[x]/x^{m+n+1}).$$

In concrete terms, this is just given by the map of commutative rings $\mathbf{F}_2[x]/(x^{m+n+1}) \to \mathbf{F}_2[x, x']/(x^{m+1}, x'^{n+1})$ given by $x \mapsto x + x'$. Passing to the limit as m and n grow, we get a map of formal schemes

$$\operatorname{Spf} \mathbf{F}_2[[x]] \times_{\operatorname{Spec} \mathbf{F}_2} \operatorname{Spf} \mathbf{F}_2[[x]] \to \operatorname{Spf} \mathbf{F}_2[[x]].$$

This map encodes a formal group law over the ring $\mathbf{F}_2$, which is given by the power series $f(x, y) = x + y \in \mathbf{F}_2[[x, y]]$.

By construction, the action of $\mathbb{G}$ on $\operatorname{Spf} \mathbf{F}_2[[x]]$ preserves the group structure given by $f(x, y) = x + y$. That is, we can regard $\mathbb{G}$ as acting by *automorphisms* of the formal group law f. This gives a description of $\mathbb{G}$ which is very convenient for our purposes:

Theorem 6. *For every commutative $\mathbf{F}_2$-algebra A, the above construction yields a canonical bijection of* $\operatorname{Hom}(\mathcal{A}^\vee, A) \simeq \operatorname{Hom}(\operatorname{Spec} A, \mathbb{G})$ *with the group of all power series*

$$x \mapsto x + a_1 x^2 + a_2 x^4 + a_3 x^8 + \ldots,$$

where $a_i \in A$, regarded as automorphisms of the formal group $\operatorname{Spec} A \times_{\operatorname{Spec} F_2} \operatorname{Spf} \mathbf{F}_2[[x]] = \operatorname{Spf} A[[x]]$.