MU and Complex Orientations (Lecture 6)

February 4, 2010

In the last lecture, we defined spectra $\mathrm{MU}(n) = \Sigma^{\infty-2n} BU(n)/BU(n-1)$ which form a direct system

$$\mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}(2) \to \cdots$$

The (homotopy) colimit of this sequence is called the *complex bordism spectrum* and is denoted by MU.

Example 1. The spectrum MU(0) is equivalent to the sphere spectrum.

Example 2. The spectrum MU(1) is the desuspension $\Sigma^{\infty-2}\, \mathbf{C}\mathrm{P}^\infty$ of $\mathbf{C}\mathrm{P}^\infty = \mathrm{BU}(1)$.

Remark 3. In terms of the above identifications, the inclusion $\mathrm{MU}(0) \to \mathrm{MU}(1)$ is given by

$$\mathrm{MU}(0) \simeq S \simeq \Sigma^{\infty-2} S^2 \to \Sigma^{\infty-2}\, \mathbf{C}\mathrm{P}^\infty = \mathrm{MU}(1).$$

Remark 4. The direct sum of complex vector bundles is classified by a multiplication $m_{a,b} : BU(a) \times BU(b) \to BU(a+b)$. Passing to Thom spectra, we get a multiplication $\mathrm{MU}(a) \otimes \mathrm{MU}(b) \to \mathrm{MU}(a+b)$. We note that the inclusion $\mathrm{MU}(n) \to \mathrm{MU}(n+1)$ can be identified with the map

$$\mathrm{MU}(n) \simeq S \otimes \mathrm{MU}(n) = \mathrm{MU}(0) \otimes \mathrm{MU}(n) \to \mathrm{MU}(1) \otimes \mathrm{MU}(n) \to \mathrm{MU}(n+1).$$

Remark 5. Taking the limit in a and b, we get a multiplication $\mathrm{MU} \otimes \mathrm{MU} \to \mathrm{MU}$. That is, MU has the structure of a ring spectrum. In fact, this multiplication is commutative and associative up to homotopy. It has a unit, given by the map $S \simeq \mathrm{MU}(0) \to \mathrm{MU}$.

In fact, the situation is much better: the multiplication on MU is commutative and associative up to all higher homotopies. That is, MU has the structure of an E_∞-ring spectrum.

Let E be a complex-oriented cohomology theory. In the last lecture, we saw that every complex vector bundle is E-orientable. In fact, for each integer n we can write down a canonical orientation of the universal bundle ζ_n of rank n on the classifying space $BU(n)$: it is classified by a map $\phi_n : \mathrm{MU}(n) \to E$. These maps ϕ_n are uniquely characterized by the following requirements:

(1) The map $\phi_1 : \mathrm{MU}(1) \to E$ is given by the complex orientation of E (note that we can identify the set of homotopy classes of maps $[\mathrm{MU}(1), E]$ with $\widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$).

(2) The maps ϕ_n are multiplicative in the following sense: for every pair of integers m and n, the diagram

$$\begin{array}{ccc} \mathrm{MU}(m) \otimes \mathrm{MU}(n) & \longrightarrow & \mathrm{MU}(m+n) \\ \downarrow{\scriptstyle \phi_m \otimes \phi_n} & & \downarrow{\scriptstyle \phi_{m+n}} \\ E \otimes E & \longrightarrow & E \end{array}$$

commutes up to homotopy.

To prove assertion (2), we recall that $E^*(\mathrm{MU}(m+n))$ can be identified (up to a shift in grading) with the ideal in $E^*(\mathrm{BU}(m+n)) \simeq (\pi_* E)[[c_1, \ldots, c_{m+n}]]$ generated by the Chern class c_{m+n}. Similarly, $E^*(\mathrm{MU}(m) \otimes \mathrm{MU}(n))$ can be identified with the ideal in $E^*(\mathrm{BU}(m) \times \mathrm{BU}(n)) \simeq (\pi_* E)[[c_1, \ldots, c_m, c'_1, \ldots, c'_n]]$ generated by the product $c_m c'_n$. The commutativity of the diagram now follows from the equation $c_{m+n}(\zeta \oplus \zeta') = c_m(\zeta) c_n(\zeta')$, where ζ and ζ' are vector bundles of rank m and n.

We claim that the composite map

$$\mathrm{MU}(n) \to \mathrm{MU}(n+1) \stackrel{\phi_{n+1}}{\to} E$$

coincides with ϕ_n. Using (2) and Remark 4, we deduce that this composite map is given by

$$\mathrm{MU}(n) \simeq \mathrm{MU}(n) \otimes \mathrm{MU}(0) \to \mathrm{MU}(n) \otimes \mathrm{MU}(1) \stackrel{\phi_n \phi_1}{\to} E.$$

We are therefore reduced to proving that $\phi_1 | \mathrm{MU}(0)$ coincides with ϕ_0 (which is the unit map $S \to E$). According to Remark 3, this map is given by the class in $\pi_0 E$ given by restricting our complex orientation $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ to $\widetilde{E}^2(S^2) \simeq \pi_0 E$, which we have assumed to be the unit in E.

The mapping spectrum E^{MU} can be obtained as a homotopy limit of mapping spectra $E^{\mathrm{MU}(n)}$. We therefore have a Milnor long exact sequence

$$\varprojlim{}^1 \{E^{-1}(\mathrm{MU}(n))\} \to E^0(\mathrm{MU}) \to \varprojlim E^0(\mathrm{MU}(n)) \to \varprojlim{}^1 \{E^0(\mathrm{MU}(n))\}.$$

The outer groups vanish, since each of the restriction maps $E^*(\mathrm{MU}(n+1)) \to E^*(\mathrm{MU}(n))$ is surjective (it corresponds, under our choice of Thom isomorphisms, to the restriction map $E^*(BU(n+1)) \to E^*(BU(n))$, which is obtained by killing c_{n+1} in the power series ring $E^*(BU(n+1)) \simeq (\pi_* E)[[c_1, \ldots, c_{n+1}]]$). It follows that the maps $\phi_n : \mathrm{MU}(n) \to E$ can be uniquely amalgamated to give a map $\phi : \mathrm{MU} \to E$.

Proposition 6. *The map ϕ is a map of ring spectra.*

Proof. We must show that the diagram

$$\begin{array}{ccc} \mathrm{MU} \otimes \mathrm{MU} & \xrightarrow{\phi \otimes \phi} & \mathrm{MU} \\ \downarrow & & \downarrow \phi \\ E \otimes E & \longrightarrow & E \end{array}$$

commutes. Repeating the above argument, we conclude that $E^0(\mathrm{MU} \otimes \mathrm{MU})$ can be obtained as the inverse limit of the cohomology groups $E^0(\mathrm{MU}(a) \otimes \mathrm{MU}(b))$. The desired result now follows from the commutativity of each of the squares

$$\begin{array}{ccc} \mathrm{MU}(a) \otimes \mathrm{MU}(b) & \longrightarrow & \mathrm{MU}(a+b) \\ \downarrow & & \downarrow \\ E \otimes E & \longrightarrow & E \end{array}$$

(see Remark 4). □

The inclusion $\Sigma^{\infty-2}\, \mathbf{C}\mathrm{P}^\infty \simeq \mathrm{MU}(1) \to \mathrm{MU}$ determines a class $t \in \widetilde{\mathrm{MU}}^2(\mathbf{C}\mathrm{P}^\infty)$. By construction, the ring spectrum map $\phi : \mathrm{MU} \to E$ carries t to our chosen complex orientation of E.

Remark 7. The class $t \in \widetilde{\mathrm{MU}}^2(\mathbf{C}\mathrm{P}^\infty)$ is a complex orientation of MU. To see this, we note that the restriction of t to $\widetilde{\mathrm{MU}}^2(S^2) \simeq \pi_0\, \mathrm{MU}$ is given by the map $S \simeq \mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}$, which is the unit for the ring spectrum MU.

We can summarize our discussion as follows:

Theorem 8. *Let E be a commutative ring spectrum, and let $t \in \widetilde{\mathrm{MU}}^2(\mathbf{CP}^\infty)$ be the complex orientation described above. The construction $(\phi : \mathrm{MU} \to E) \mapsto \phi(t)$ determines a bijection between complex orientations of E and ring spectrum maps $\mathrm{MU} \to E$.*

In other words, the complex bordism spectrum MU is the *universal* complex-oriented cohomology theory.

Proof. The above analysis shows that given any complex orientation of E, we can construct a ring spectrum map $\phi : \mathrm{MU} \to E$ which carries our complex orientation to the specified complex orientation of E. It remains to prove injectivity. Let $\phi, \phi' : \mathrm{MU} \to E$ be two ring spectrum maps which determine the same complex orientation of E; we wish to prove that ϕ and ϕ' are homotopic. The condition that ϕ and ϕ' determine the same complex orientation tells us that $\phi| \mathrm{MU}(1) \simeq \phi'| \mathrm{MU}(1)$. Since E is complex-orientable, the preceding argument shows that $E^0(\mathrm{MU}) \simeq \varprojlim E^0(\mathrm{MU}(n))$. It will therefore suffice to show that ϕ and ϕ' have the same restriction to $\mathrm{MU}(n)$ for every integer n. Since ϕ is a ring map, the composition

$$\mathrm{MU}(1)^{\otimes n} \to \mathrm{MU}(n) \xrightarrow{\phi} E$$

is a product of n copies of $\phi| \mathrm{MU}(1)$ in $E^0(\mathrm{MU}(1))$, and therefore coincides with the composition

$$\mathrm{MU}(1)^{\otimes n} \to \mathrm{MU}(n) \xrightarrow{\phi'} E.$$

It will therefore suffice to show that the restriction maps $E^0(\mathrm{MU}(n)) \to E^0(\mathrm{MU}(1)^{\otimes n})$. Using our Thom isomorphisms (provided by any complex orientation of E), this is equivalent to the injectivity of the map $E^0(\mathrm{BU}(n)) \to E^0(\mathrm{BU}(1)^n)$, which is the "spliting principle" we discussed earlier. □

Theorem 8 suggests that if we are interested in complex-oriented cohomology theories and the associated formal group laws, then we should focus our attention on the complex bordism spectrum MU. The universal complex orientation determines a (graded) formal group law $f(x, y) \in (\pi_* \mathrm{MU})[[x, y]]$. As we have seen, this formal group law is given by a map of graded rings $L \to \pi_* \mathrm{MU}$.

Our goal next week will be to prove the following theorem:

Theorem 9 (Quillen)**.** *The map $L \to \pi_* \mathrm{MU}$ is an isomorphism. (In particular, the spectrum* MU *has homotopy groups only in even degrees.)*