Complex Bordism (Lecture 5)

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In this lecture, we will introduce another important example of a complex-oriented cohomology theory: the cohomology theory MU of *complex bordism*. In fact, we will show that MU is *universal* among complex-oriented cohomology theories.

We begin with a general discussion of orientations. Let X be a topological space and let ζ be a vector bundle of rank n on X. We may assume without loss of generality that ζ is equipped with a metric, so that the unit ball bundle $B(\zeta) \to X$ and the unit sphere bundle $S(\zeta) \to X$ are well-defined. If E is an arbitrary cohomology theory, we can define the twisted E-cohomology $E^{*-\zeta}(X)$ to be the relative cohomology $E^*(B(\zeta), S(\zeta))$.

Example 1. If ζ is the trivial bundle of rank n, then $B(\zeta) \simeq B^n \times X$ and $S(\zeta) \simeq S^{n-1} \times X$. In this case, we have a canonical isomorphism $E^{*-\zeta}(X) = E^*(X \times B^n, X \times S^{n-1}) \simeq E^{*-n}(X)$.

If E is a multiplicative cohomology theory, then $E^{*-\zeta}(X)$ is a module over $E^*(X)$.

Definition 2. Let ζ be a vector bundle of rank n on a space X, and let E be a multiplicative cohomology theory. An *E-orientation of* ζ is a cohomology class $u \in E^{n-\zeta}(X) \simeq E^n(B(\zeta), S(\zeta))$ such that:

- $(*)$ For every point $x \in X$, the restriction $x^*(u) \in E^{n-\zeta_x}(\{x\}) \simeq E^0(\{x\})$ is a generator of $E^*(\{x\}) \simeq \pi_* E$ (as a $\pi_* E$-module).

In this case, we say that u is a *Thom class* for ζ in E-cohomology.

Remark 3. The identification $E^{n-\zeta_x}(\{x\}) \simeq E^0(\{x\})$ is noncanonical: it depends on a trivialization of the fiber ζ_x. This dependence is not very strong, since the orthogonal group $O(n)$ has only two components: the resulting elements of $E^0(\{x\})$ are off by a sign if we choose trivializations with different orientations.

Remark 4. A consequence of Definition 2 is that the Leray-Serre spectral sequence for the fibration $S(\zeta) \to X$ degenerates and gives an identification $E^*(X) \simeq E^{*+n-\zeta}(X)$, given by multiplication by u.

Remark 5. In the setting of Definition 2, it suffices to check the condition at one point x in each connected component of X.

Our next goal is to show that if E is a complex-oriented cohomology theory, then all complex vector bundles have a canonical E-orientation. To prove this, it suffices to consider the universal case: that is, the case of a universal bundle ζ of (complex) rank n over the classifying space $BU(n)$. Recall that $BU(n)$ can be realized as the quotient of a contractible space $EU(n)$ by a free action of the unitary group $U(n)$. In this case, the subgroup $U(n-1)$ also acts freely on $EU(n)$, so the quotient $EU(n)/U(n-1)$ is a model for the classifying space $BU(n-1)$. This realization gives us a fibration $BU(n-1) \to BU(n)$, whose fiber is the quotient group $U(n)/U(n-1) \simeq S^{2n-1}$. In fact, this sphere bundle can be identified with the unit sphere bundle $S(\zeta)$. Since $B(\zeta) \simeq BU(n)$, we get canonical isomorphisms $E^{*-\zeta}(BU(n)) = E^*(BU(n), BU(n-1))$.

We have computed these groups: the group $E^*(BU(n))$ is a power series ring $(\pi_* E)[[c_1, \ldots, c_n]]$ and the group $E^*(BU(n-1))$ is a power series ring $(\pi_* E)[[c_1, \ldots, c_{n-1}]]$. The restriction map $\theta : E^*(BU(n)) \to E^*(BU(n-1))$ is a surjective ring homomorphism. It follows that the relative cohomology group $E^*(BU(n), BU(n-1))$ can be identified with the kernel of θ: that is, with the ideal $c_n(\pi_* E)[[c_1, \ldots, c_n]]$. This is in fact a free module over $E^*(BU(n))$, which suggests the following:

Proposition 6. *The cohomology class $c_n \in E^{2n}(BU(n), BU(n-1)) \simeq E^{2n-\zeta}(BU(n))$ is a Thom class (so that the universal bundle ζ on $BU(n)$ has a canonical E-orientation).*

Proof. We must check that condition $(*)$ holds at every point of $BU(n)$. Since $BU(n)$ is connected, it will suffice to check $(*)$ at *any* points of $BU(n)$. We may therefore replace ζ by its pullback along the map $f : BU(1)^n \to \mathrm{BU}(n)$ and c_n by its image in

$$E^{*-f^*\zeta}(BU(1)^n) \simeq (t_1 \dots t_n)(\pi_* E)[[t_1, \dots, t_n]] \subseteq (\pi_* E)[[t_1, \dots, t_n]] \simeq E^*(BU(1)^n),$$

which can be identified with the product $t_1 \dots t_n$. Since $f^*\zeta$ is a direct sum $\oplus_{1 \leq i \leq n} p_i^* \mathcal{O}(1)$ of pullbacks of the universal line bundle $\mathcal{O}(1)$ on $BU(1) \simeq \mathbf{C}\mathrm{P}^\infty$ along the projection maps $p_i : BU(1)^n \to BU(1)$, we can reduce to proving the assertion in the case $n = 1$. In this case, $E^{*-\zeta}(BU(1))$ can be identified with the reduced cohomology $\widetilde{E}^*(\mathbf{C}\mathrm{P}^\infty)$, and the condition that $u \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ be an orientation of $\mathcal{O}(1)$ is that it maps to a unit when restricted to $\widetilde{E}^2(S^2) \simeq \pi_0 E$. Our complex orientation is even better: it maps to $1 \in \pi_0 E$. □

If ζ' is any complex vector bundle of rank n on any (nice) space X, then we can write $\zeta' = f^*\zeta$ for some classifying map $f : X \to BU(n)$. We can then define an orientation $u_{\zeta'} \in E^{2n-\zeta'}(X)$ to be the pullback of $c_n \in E^{2n-\zeta}(BU(n))$.

By construction, our Chern classes in E-cohomology have the same behavior with respect to direct sums of vector bundles as the usual Chern classes: namely, we have

$$c_n(\zeta \oplus \zeta') = \sum_{i+j=n} c_i(\zeta) c_j(\zeta').$$

In particular, if ζ and ζ' have ranks a and b, then we have $c_{a+b}(\zeta + \zeta') = c_a(\zeta) c_b(\zeta')$. We conclude from this:

(1) If ζ and ζ' are complex vector bundles of rank a and b on a space X, then the Thom classes $u_\zeta \in E^{2a-\zeta}(X)$ and $u_{\zeta'} \in E^{2b-\zeta'}(X)$ have product $u_{\zeta+\zeta'} \in E^{2a+2b-(\zeta \oplus \zeta')}(X)$.

We also have the following

(2) Let ζ be the trivial bundle of rank 1 on a space X. Then the Thom class $u_\zeta \in E^{2-\zeta}(X) \simeq E^0(X)$ coincides with the unit. This is a translation of our assumption that $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ restricts to the unit in $\widetilde{E}^2(S^2) \simeq \pi_0 E$.

Definition 7. For each integer n, we let $\mathrm{MU}(n)$ denote the Thom spectrum $\Sigma^{-2n} BU(n)^{\zeta_n} = \Sigma_+^{\infty-2n} BU(n)/BU(n-1)$, where ζ_n denotes the universal bundle of rank n. The restriction of ζ_n to $BU(n-1)$ is the sum of a trivial bundle $\mathbf{1}$ of rank 1 with a bundle ζ_{n-1}. We therefore have a canonical map

$$\mathrm{MU}(n-1) \simeq \Sigma^{2-2n} BU(n-1)^{\zeta_{n-1}} = \Sigma^{-2n} BU(n-1)^{\zeta_{n-1} \oplus \mathbf{1}} \to \Sigma^{-2n} BU(n)^{\zeta_n} = \mathrm{MU}(n).$$

The universal Thom class $c_n \in E^n(BU(n)/BU(n-1))$ can be interpreted as a map of spectra $\phi_n : \mathrm{MU}(n) \to E$. It follows from (1) and (2) that the restriction of this map to $\mathrm{MU}(n-1)$ is homotopic to ϕ_{n-1}. In the next lecture, we will see that the maps $\{\phi_n\}_{n \geq 0}$ therefore determine a map from the colimit

$$S \simeq \mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}(2) \to \dots$$

into E.

Definition 8. The colimit $\varinjlim \mathrm{MU}(n)$ is denoted by MU; it is called the *complex bordism spectum*.

Remark 9. The complex bordism spectrum MU can be described as a Thom spectrum for the space $BU = \varinjlim BU(n)$. However, it is not a Thom spectrum for a vector bundle of any particular rank: rather, it is the Thom spectrum for a virtual bundle of rank 0, whose restriction to each $BU(n)$ is a formal difference $\zeta_n - \mathbf{1}^n$.

Remark 10. The complex bordism spectrum has a natural geometric interpretation. Namely, each homotopy group $\pi_n E$ can be identified with the group of bordism classes of n-dimensional manifolds M equipped with a stable almost complex structure (that is, a complex structure on the direct sum of the tangent bundle M with a trivial vector bundle of sufficiently large rank). More generally, if X is any space, we can identify the homology groups $E_n X$ with bordism groups of stably almost complex n-manifolds equipped with a map to X.