The Chromatic Convergence Theorem (Lecture 32)

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Fix a prime number p. For any p-local spectrum X, one can arrange its $E(n)$-localizations into the *chromatic tower*

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

Our goal in this lecture and the next is to prove the following result:

Theorem 1 (Chromatic Convergence)**.** *If X is a finite p-local spectrum, then X is a homotopy limit of its chromatic tower.*

Remark 2. The collection of p-local spectra which satisfy the conclusion of Theorem 1 is obviously thick. It therefore suffices to prove Theorem 1 for a single p-local spectrum of type 0: for example, the p-local sphere).

For every spectrum X, let $C_n(X)$ denote the homotopy fiber of the map $X \to L_{E(n)}X$. Then $\varprojlim C_n(X)$ is the homotopy fiber of the map $X \to \varprojlim L_{E(n)}X$. The chromatic convergence theorem is therefore equivalent to the following:

Theorem 3. *The homotopy limit of the tower $\{C_n(S_{(p)})\}$ is trivial. Even better: for every integer m, the tower of abelian groups $\{\pi_m C_n(S_{(p)})\}$ is trivial (as a pro-abelian group).*

The starting point for Theorem 3 is the following result, which we will prove in the next lecture:

Proposition 4. *Each of the maps $C_n(S_{(p)}) \to C_{n-1}(S_{(p)})$ induces the zero map $\mathrm{MU}_*(C_n(S_{(p)})) \to \mathrm{MU}_*(C_{n-1}(S_{(p)})$.*

Let us assume Proposition 4 and see how it leads to a proof of Theorem 3. To this end, we recall the definition of the *Adams-Novikov filtration* on the homotopy groups π_*X of a spectrum X. Let I denote the homotopy fiber of the unit map $S \to \mathrm{MU}$. There is an evident map $I \to S$, which induces a map $I^{\otimes m} \to S$ for each m. We say that an element $x \in \pi_n X$ has *Adams-Novikov filtration* $\geq m$ if x lies in the image of the map $\pi_n(I^{\otimes m} \otimes X) \to \pi_n X$.

Lemma 5. *Let $f : X \to Y$ be a map of spectra such that f induces the zero map $\theta : \mathrm{MU}_*(X) \to \mathrm{MU}_*(Y)$. Then f increases Adams-Novikov filtration. That is, if $x \in \pi_n X$ has Adams-Novikov filtration $\geq m$, then $f(x) \in \pi_n Y$ has Adams-Novikov filtration $\geq m+1$.*

Proof. Lift x to a class $\overline{x} \in \pi_n(I^{\otimes m} \otimes X)$. We then obtain $f(\overline{x}) \in \pi_n(I^{\otimes m} \otimes Y)$ lifting y. To lift y to $\pi_n(I^{\otimes m+1} \otimes Y)$, it suffices to show that the image of $\overline{y}$ vanishes in $I^{\otimes m} \otimes Y \otimes \mathrm{MU}$. Consequently, it will suffice to show that f induces the zero map

$$\theta_m : \mathrm{MU}_*(I^{\otimes m} \otimes X) \to \mathrm{MU}_*(I^{\otimes m} \otimes Y).$$

Recall that $\mathrm{MU}_*(\mathrm{MU}) \simeq (\pi_* \mathrm{MU})[b_1, b_2, \ldots]$ is a free $\pi_* \mathrm{MU}$-module on a basis consisting of monomials in the b_i. It follows that $\mathrm{MU}_*(\Sigma I)$ is a free $\pi_* \mathrm{MU}$-module on a basis consisting of monomials of positive length in the b_i. In particular, $\mathrm{MU} \otimes I$ is a free module over MU, so we have Kunneth decompositions

$$\mathrm{MU}_*(I^{\otimes m} \otimes X) = \mathrm{MU}_*(I)^{\otimes m} \otimes_{\pi_* \mathrm{MU}} \mathrm{MU}_*(X)$$

$$\mathrm{MU}_*(I^{\otimes m} \otimes Y) = \mathrm{MU}_*(I)^{\otimes m} \otimes_{\pi_* \mathrm{MU}} \mathrm{MU}_*(Y)$$

Since $\theta = 0$, it follows that $\theta_m = 0$. □

Combining Lemma 5 with Proposition 4, we deduce:

Proposition 6. *For all m, n, and s, the image of the map*

$$\pi_n C_{m+s} S_{(p)} \to \pi_n C_m S_{(p)}$$

consists of elements having Adams-Novikov filtration $\geq s$.

To complete the proof of Theorem 3, it will suffice to show the following:

Proposition 7. *For every pair of integers m and n, the Adams-Novikov filtration on $\pi_n C_m(S_{(p)})$ is finite. That is, there exists an integer s such that every element $x \in \pi_n C_m(S_{(p)})$ of Adams-Novikov filtration $\geq s$ is trivial.*

Let us now introduce some terminology which will be useful for proving Proposition 7.

Definition 8. Let $f : X \to Y$ be a map of spectra. We say that f is *phantom below dimension n* if the following condition is satisfied: for every finite spectrum F of dimension $\leq n$ and every map $u : F \to X$, the composition $f \circ u$ is nullhomotopic.

Remark 9. The map f is phantom if and only if it is phantom below dimension n, for every integer n.

Definition 10. A spectrum X is MU-*convergent* if, for every integer n, there exists s such that the map $I^{\otimes s} \otimes X \to X$ is phantom below dimension n.

If X is MU-convergent and n, s are as in Definition 10, then the map $I^{\otimes s} \otimes X \to X$ is trivial on π_n and so every element of $\pi_n X$ having Adams-Novikov filtration $\geq s$ is zero. Proposition 7 is therefore a consequence of the following:

Proposition 11. *Let X be any connective spectrum. Then $C_m(X)$ is* MU-*convergent for each $m \geq 0$.*

We need a few preliminary observations.

Lemma 12. *Let $f : X \to Y$ phantom below dimension n, and let W be a connective spectrum. Then the induced map $X \otimes W \to Y \otimes W$ is phantom below dimension n.*

Proof. Let F be a finite spectrum of dimension $\leq n$ and consider a map $u : F \to X \otimes W$. We wish to prove that $(f \otimes \mathrm{id}_W) \circ u$ is nullhomotopic. We can write W as a filtered colimit of finite connective spectra W_α. Since F is finite, u factors through $X \otimes W_\alpha$ for some α. Replacing W by W_α, we may assume that W is finite. In this case, we can identify u with a map $v : DW \otimes F \to X$. Since W is connective, $DW \otimes F$ has dimension $\leq n$; it follows that $f \circ v$ is nullhomotopic so that $(f \otimes \mathrm{id}_W) \circ u$ is nullhomotopic. □

Lemma 13. *Suppose we are given a fiber sequence of spectra*

$$X \to Y \to Z.$$

If X and Z are MU-*convergent, then Y is* MU-*convergent.*

Proof. Fix an integer n, and choose s such that the maps $I^{\otimes s} \otimes X \to X$ and $K^{\otimes s} \otimes Z \to Z$ are phantom below n. We will show that the map $I^{\otimes 2s} \otimes Y \to Y$ is phantom below n. Let F be a finite spectrum of dimension $\leq n$ with a map $u : F \to I^{\otimes 2s} \otimes Y$. Since $I^{\otimes 2s} \otimes Z \to I^{\otimes s} \otimes Z$ is phantom below n (Lemma 12), the composite map

$$F \to I^{\otimes 2s} \otimes Y \to I^{\otimes 2s} \otimes Z \to I^{\otimes s} \otimes Z$$

is nullhomotopic. It follows that the composition

$$F \otimes I^{\otimes 2s} \otimes Y \to I^{\otimes s} \otimes Y$$

factors through some map $v : F \to I^{\otimes s} \otimes X$. Then the composition

$$F \xrightarrow{u} I^{\otimes 2s} \otimes Y \to Y$$

is given by

$$F \xrightarrow{v} I^{\otimes s} \otimes X \to X \to Y$$

and is therefore nullhomotopic. □

Lemma 14. *Let X be an* MU*-module spectrum. Then X is* MU*-convergent.*

Proof. The unit map $X \to \mathrm{MU} \otimes X$ admits a section, given by the action of $\mathrm{MU}_{(p)}$ on X. This is equivalent to the statement that the map $I \otimes X \to X$ is nullhomotopic (and hence phantom below n, for any n). □

Lemma 15. *Let X be any spectrum. For each $n \geq 0$, the spectrum $L_{E(n)}X$ is* MU*-convergent.*

Proof. Let $X^\bullet = E(n)^{\otimes(\bullet+1)} \otimes X$ and let $\{\mathrm{Tot}^m X^\bullet\}$ be the $E(n)$-based Adams tower of X. The proof of the smash product theorem shows that $\{\mathrm{Tot}^m X^\bullet\}$ is equivalent to the constant tower with value $L_{E(n)}X$. It follows that $L_{E(n)}X$ is a retract of $\mathrm{Tot}^m X^\bullet$ for some m. It therefore suffices to show that each $\mathrm{Tot}^m X^\bullet$ is MU-convergent. Each $\mathrm{Tot}^m X^\bullet$ is a finite homotopy inverse limit of the spectra X^k; by Lemma 13 it suffices to show that each X^k is MU-convergent. But $X^k \simeq E(n)^{\otimes k+1} \otimes X$ has the structure of an $E(n)$-module spectrum. Since $E(n)$ is complex orientable, there is a map of ring spectra $\mathrm{MU} \to E(n)$ so that X^k admits an MU-module structure; the desired result now follows from Lemma 14. □

Lemma 16. *Let X be a connective spectrum. Then X is* MU*-convergent.*

Proof. We claim that for any finite CW complex F of dimension $\leq n$ and any map $u : F \to I^{\otimes n+1} \otimes X$, the composite map $u : F \to I^{\otimes n+1} \otimes X \to X$ is nullhomotopic. In fact, u itself is nullhomotopic, because $I^{\otimes n+1} \otimes X$ is n-connected. To check this, we note that since X is connective it suffices to show that K is connected: that is, we have $\pi_i K \simeq 0$ for $i \leq 0$. This follows from the long exact sequence associated to the fiber sequence

$$I \to S \to \mathrm{MU},$$

since the map $\pi_i S \to \mathrm{MU}$ is bijective for $i \leq 0$ and surjective when $i = 1$. □

Proof of Proposition 11. Let X be a connective spectrum. We have a fiber sequence

$$C_n(X) \to X \to L_{E(n)}X$$

where X is MU-convergent by Lemma 16 and $L_{E(n)}(X)$ is MU-convergent by Lemma 15. It follows from Lemma 13 that $C_n(X)$ is MU-convergent. □