The Smash Product Theorem (Lecture 31)

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In this lecture, we will apply the results of Lecture 30 to prove (part of) the smash product theorem. We begin by summarizing the results of the previous lecture. Fix a ring spectrum E (which we take to be a structured ring spectrum for simplicity). For every specturm X, we let $X^{\bullet}$ denote the cosimplicial spectrum $[n] \mapsto E^{\otimes n+1} \otimes X$.

Proposition 1. *Let E be a ring spectrum and X an arbitrary spectrum. Suppose that there exists an integer $s \geq 1$ such that, for every finite spectrum F, the E-based Adams spectral sequence $\{E_r^{p,q}, d_r\}$ for $X \otimes F$ has $E_s^{p,q} \simeq 0$ for $p \geq s$. Then the Adams tower $\{\mathrm{Tot}^n X^{\bullet}\}$ is equivalent, as a pro-object, to a constant tower.*

In the situation of Proposition 1, we have an equivalence of pro-spectra $\{\mathrm{Tot}^n X^{\bullet}\} \simeq X'$ for some spectrum X'. We saw in the last lecture that $X' \simeq \mathrm{Tot}(X^{\bullet})$ can be identified with the localization $L_E X$. Note that if the hypotheses of Proposition 1 are satisfied, then for every other spectrum Y, the tower $\{\mathrm{Tot}^n (X \otimes Y)^{\bullet}\} \simeq \{\mathrm{Tot}^n X^{\bullet} \otimes Y\}$ is pro-equivalent to the constant tower with value $X' \otimes Y$. It follows that the canonical map

$$(L_E X) \otimes Y \rightarrow L_E(X \otimes Y)$$

is a homotopy equivalence.

Proposition 2. *Let E be a p-local ring spectrum, and suppose that there exists a finite p-local spectrum X of type 0 which satisfies the hypotheses of Proposition 1. Then L_E is a smashing localization.*

Proof. Let $\mathcal{T}$ be the collection of p-local finite spectra X such that, for every p-local spectrum Y, the canonical map $(L_E X) \otimes Y \rightarrow L_E(X \otimes Y)$ is an equivalence. It is clear that $\mathcal{T}$ is thick. If $\mathcal{T}$ contains a finite p-local spectrum of type 0, then the thick subcategory theorem implies that $\mathcal{T}$ contains every finite p-local spectrum; in particular, $S_{(p)} \in \mathcal{T}$ so that $L_E S_{(p)} \otimes Y \simeq L_E(Y)$ for all Y and therefore L_E is smashing. □

Let us now specialize to the case where E is Morava E-theory $E(n)$. The spectrum $E(n)$ is complex-oriented, and the map $\mathrm{Spec}\, \pi_* E \rightarrow \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{Z}_{(p)}$ is a faithfully flat cover of the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ classifying formal groups of height $\leq n$. For every spectrum X, let $\mathcal{F}_X$ denote the associated quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}$. The E_2-term of the $E(n)$-based Adams-Novikov spectral sequence for X is given by the cohomology of the chain complex

$$E(n)_*(X) \rightarrow (E(n) \otimes E(n))_* X \rightarrow (E(n) \otimes E(n) \otimes E(n))_* X \rightarrow \cdots,$$

which computes the cohomology of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ with coefficients in the quasi-coherent sheaves $\mathcal{F}_{\Sigma^k X}$ for varying k. Consequently, we obtain the following:

Proposition 3. *Suppose that there exists a finite p-local spectrum X of type 0 and an integer $s_0 \geq 1$ with the following property: for every finite spectrum F, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\leq n}; \mathcal{F}_{X \otimes F})$ vanish for $s \geq s_0$. Then the localization $L_{E(n)}$ is smashing.*

To make things more concrete, let us assume that X is an *even* finite p-local spectrum: that is, a finite p-local spectrum whose homology groups $\mathrm{H}_*(X; \mathbf{Z}_{(p)})$ are free $\mathbf{Z}_{(p)}$-modules concentrated in even degrees. This is equivalent to saying that X admits a finite cell decomposition, where each cell is an even suspension of $S_{(p)}$. Such a spectrum is always of type 0, provided that it is nonzero. For such a spectrum, the Atiyah-Hirzebrucn spectral sequence for computing $(\mathrm{MU}_{(p)})_*(X)$ degenerates: that is, $\mathrm{MU}_{(p)} \otimes X$ is a free module over $\mathrm{MU}_{(p)}$ (on generators corresponding to some basis for $\mathrm{H}_*(X; \mathbf{Z}_{(p)})$). It follows that $\mathcal{F}_X$ is a vector bundle on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$, and that for any other spectrum F we have

$$(\mathrm{MU}_p)_*(X \otimes F) \simeq \pi_*((\mathrm{MU}_{(p)} \otimes X) \otimes_{\mathrm{MU}_{(p)}} (\mathrm{MU}_{(p)} \otimes F)) \simeq (\mathrm{MU}_{(p)})_*(X) \otimes_{\mathrm{MU}_{(p)}} (\mathrm{MU}_{(p)})_* F.$$

On the moduli stack $\mathcal{M}_{\mathrm{FG}}$, we deduce that the canonical map of quasi-coherent sheaves

$$\mathcal{F}_X \otimes \mathcal{F}_F \to \mathcal{F}_{X \otimes F}$$

is an isomorphism after localization at p. We conclude the following:

Proposition 4. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^{\leq n}$, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\leq n}) \otimes \mathcal{G})$ vanish for $s \geq s_0$. Then the localization functor $L_{E(n)}$ is smashing.*

We can attack this problem using the filtration of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ by height. Suppose we have chosen an even finite p-local spectrum X. For each $k \leq n$, let $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$ denote the closed substack of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ classifying formal groups which have height $\geq k$. Let us attempt to prove that, for every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$, the groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G})$ vanish for large s. The idea is to use descending induction on k. Note that $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$ can be regarded as a closed substack of $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$: it is the zero locus of v_k, which we regard as a section of $\omega^{p^k - 1}$ (here ω is the line bundle on $\mathcal{M}_{\mathrm{FG}}$ given by assigning to each formal group the dual of its Lie algebra). In particular, multiplication by v_n induces a map of sheaves

$$\mathcal{G} \to \mathcal{G} \otimes \omega^{p^k - 1}$$

whose kernel and cokernel are supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$. We may therefore assume, by our inductive hypothesis, that v_n induces an isomorphism

$$\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G}) \to \mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G} \otimes \omega^{p^k - 1})$$

for sufficiently large s. It follows that for large s, we have an isomorphism

$$\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G}) \simeq \varinjlim_m \mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G} \otimes \omega^{(p^k - 1)m});$$

here the latter group can be identified with the cohomology of $\mathcal{F}_X \otimes \mathcal{G}$ on the open substack of $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$ complementary to $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$: this is the moduli stack of formal groups of height exactly k. We are therefore reduced to the following:

Proposition 5. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for $0 \leq k \leq n$ and every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^{k}$, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{k}; (\mathcal{F}_X \mid \mathcal{M}_{\mathrm{FG}}^{k}) \otimes \mathcal{G})$ vanish for $s \geq s_0$.*

The vanishing condition appearing in Proposition 6 is automatic when $k = 0$, since quasi-coherent sheaves on $\mathcal{M}_{\mathrm{FG}}^0 \simeq B\mathbb{G}_m$ have no higher cohomology. Assume that $k > 0$, and choose a formal group law $f(x, y)$ of height k over $\overline{\mathbf{F}}_p$. Let G_k denote the automorphism group of f (as a formal group law over $\overline{\mathbf{F}}_p$), regarded as a profinite group. We have a pullback diagram of algebraic stacks

$$\begin{array}{ccc} BG_k \times \operatorname{Spec} \overline{\mathbf{F}}_p & \longrightarrow & \mathcal{M}_{\mathrm{FG}}^k \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{\mathbf{F}}_p & \longrightarrow & \operatorname{Spec} \mathbf{F}_p. \end{array}$$

This implies:

(a) Every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}_{\mathrm{FG}}^k$ determines an $\overline{\mathbf{F}}_p$-vector space V equipped with a continuous action of G_k.

(b) We have a canonical isomorphism $\mathrm{H}^*(\mathcal{M}_{\mathrm{FG}}^k; \mathcal{F}) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \simeq \mathrm{H}^*(G_k; V)$.

Consequently, we are reduced to proving the following:

Proposition 6. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for $1 \leq k \leq n$, if we let V denote the representation of the profinite group G_k associated to $\mathcal{F}_X$, then $\mathrm{H}^s(G_k; V \otimes W) \simeq 0$ for $s \geq s_0$ and any continuous representation W of G_k. Then $L_{E(n)}$ is a smashing localization.*

Let us now indicate briefly why it is plausible that the hypothesis of Proposition 6 should be satisfied. Fix $1 \leq k \leq n$. Recall that G_k can be described as the group of units in the ring $\mathrm{End}(f)$, which is a noncommutative valuation ring of rank k^2 over $\mathbf{Z}_p$. In particular, it is a p-adic Lie group. Consequently, on a sufficiently small open subgroup of G_k, the group structure on G_k closely approximates the (commutative) group structure on $\mathbf{Z}_p^{k^2}$. If M is a discrete p-torsion module over $\mathbf{Z}_p^{k^2}$, then the profinite group cohomology $\mathbf{Z}_p^{k^2}$ with coefficients in M agrees with the ordinary group cohomology of $\mathbf{Z}^{k^2}$ with coefficients in M, and therefore vanishes in degrees larger than k^2. Using this, Lazard shows that there is an open subgroup $U \subseteq G_k$ such that $\mathrm{H}^s(U; M)$ vanishes for $s > k^2$ and any G_k-module M. The same result does not necessarily hold when $U = G_k$. However, an argument of Serre shows that if G_k is not of finite cohomological dimension, then it must contain an element of order p. However, elements of order p are well-understood: these are pth roots of unity in the division algebra $D = \mathrm{End}(f)[p^{-1}]$, and they exist only when the rank k of D is divisible by the degree $(p-1)$ of the field extension $\mathbf{Q}_p(\zeta_p)$. In particular, if $p > n+1$, then the profinite groups $\{G_k\}_{1 \leq k \leq n}$ have finite cohomological dimension and the hypotheses of Proposition 6 are satisfied for $X = S_{(p)}$.

When $p \leq n$, we need to work harder. In this case, some of the groups G_k do contain elements of order p. However, each G_k contains at most a single conjugacy class of subgroups V having order p: this follows from the Skolem-Noether theorem. In this case, the subgroup V can be regarded as the "obstruction" to G_k being of finite cohomological dimension: one can show that the cohomology groups $\mathrm{H}^*(G_k; M)$ are bounded if the subgroup V acts freely on M. It therefore suffices to choose a spectrum X such that the associated representation V of G_k is free over V. This requires some representation-theoretic constructions which we will not pursue further.