Telescopic vs. E_n-Localization (Lecture 29)

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Let p be a prime number, fixed throughout this lecture. Let L be a Bousfield localization functor on p-local spectra. Our goal in this lecture is to obtain a structure theorem for L, under the assumption that L is smashing.

Let us begin by fixing a bit of terminology. We say a spectrum X is *L-local* if the map $X \rightarrow LX$ is an equivalence.

Lemma 1. *Let L be a localization functor. For $0 \leq n \leq \infty$, we have either $LK(n) \simeq 0$ or $LK(n) \simeq K(n)$.*

Proof. We have a map of ring spectra $K(n) \rightarrow LK(n)$. Consequently, $LK(n)$ has the structure of a $K(n)$-module. If $LK(n) \neq 0$, then $LK(n)$ contains $K(n)$ (possibly shifted) as a retract. Since $LK(n)$ is L-local, we conclude that $K(n)$ is L-local so that $K(n) \simeq LK(n)$. □

Lemma 2. *Let L be a smashing localization functor and let E be a nonzero complex-oriented cohomology theory whose formal group has height exactly n. Then $LE \simeq 0$ if and only if $LK(n) \simeq 0$.*

Proof. If $LE \simeq 0$, then $0 \simeq K(n) \otimes LE \simeq LK(n) \otimes E$. Since $K(n) \otimes E \neq 0$, we conclude that $LK(n) \simeq 0$ (Lemma 1). Conversely, suppose that $LK(n) \simeq 0$. Then $0 \simeq LK(n) \otimes E \simeq K(n) \otimes LE$. On the other hand, $LE \otimes K(m) \simeq 0$ for $m \neq n$, since it is a complex oriented ring spectrum whose formal group has height exactly m and exactly n. It follows from the nilpotence theorem that $LE \simeq 0$. □

Lemma 3. *Let L be a smashing localization functor. If $LK(m) \simeq 0$, then $LK(n) \simeq 0$ for $n > m$.*

Proof. For $k \geq 0$, let $M(k)$ denote the cofiber of the map $t_k : \Sigma^{2k} \mathrm{MU}_{(p)} \rightarrow \mathrm{MU}_{(p)}$, and let R be the ring spectrum obtained by smashing (over $\mathrm{MU}_{(p)}$) the spectra $\{M(k)\}_{k \neq p^{m-1}, p^{n-1}}$ with $\mathrm{MU}_{(p)}[v_n^{-1}]$. For notational simplicity we will assume that $0 < m < n < \infty$, so that $\pi_* R \simeq \mathbf{F}_p[v_m, v_n^{\pm 1}]$. Note that $R[v_m^{-1}]$ is a ring spectrum whose associated formal group has height exactly m. It follows from Lemma 2 that $LR[v_m^{-1}] \simeq 0$. Since L is smashing, we can identify $LR[v_m^{-1}]$ with the colimit of the sequence

$$LR \stackrel{v_m}{\rightarrow} \Sigma^{-2(p^m-1)} LR \stackrel{v_m}{\rightarrow} \Sigma^{-4(p^m-1)} LR \rightarrow \cdots$$

It follows that $1 \in \pi_0 LR$ vanishes in $\pi_0 \Sigma^{-2k(p^m-1)} R$ for $k \gg 0$: in other words, the image of v_m^k vanishes in $\pi_* LR$. Let R' denote the cofiber of the map $v_m^{k+1} : \Sigma^{2(k+1)(p^m-1)} R \rightarrow R$, so that v_m^k vanishes in $\pi_* LR'$. Since $\pi_* R' \simeq \mathbf{F}_p[v_m, v_n^{\pm 1}]/(v_m^{k+1})$, we conclude that the map $\pi_* R' \rightarrow \pi_* LR'$ is not injective. In particular, R' is not L-local. Note that R' can be obtained as a successive extension of $k+1$ copies of $R/v_m \simeq K(n)$. It follows that $K(n)$ is not L-local. According to Lemma 1, this means that $LK(n) \simeq 0$. □

If L is any localization functor, let us denote by $\ker(L)$ the collection of all L-acyclic spectra: that is, spectra X such that $LX \simeq 0$.

Lemma 4. *Let L be a smashing localization functor, and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) $LK(n) \simeq 0$.

(2) $LK(m) \simeq 0$ *for* $n \leq m \leq \infty$.

(3) *Every finite* p*-local spectrum* X *of type* $\geq n$ *belongs to* $\ker(L)$.

(4) *There exists a finite* p*-local spectrum* X *of type* n *in* $\ker(L)$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3. The implication (3) $\Rightarrow$ (4) is clear (since there exists a finite p-local spectrum of type n). To prove that (4) $\Rightarrow$ (1), we note that $LX \simeq 0$ implies $LX \otimes K(n) \simeq X \otimes LK(n) \simeq 0$. If $LK(n) \neq 0$, then $LK(n) \simeq K(n)$ so that $X \otimes LK(n) \neq 0$, since X has type n.

It remains to prove that (2) $\Rightarrow$ (3). Let X be a p-local finite spectrum of type $\geq n$. We wish to prove that $LX \simeq 0$. Let $R = X \otimes DX$; since LX is an LR-module, it will suffice to show that $LR \simeq 0$. Since LR is a ring spectrum, by the nilpotence theorem it will suffice to show that $LR \otimes K(m) \simeq 0$ for every m. If $m < n$, we have $LR \otimes K(m) \simeq L(R \otimes K(m)) \simeq 0$ since R has type $\geq n > m$. If $m \geq n$, then $LR \otimes K(m) \simeq R \otimes LK(m) \simeq 0$ because $LK(m) \simeq 0$ by assumption (2). □

(A) We have $LK(n) \simeq 0$ for all $0 \leq n < \infty$.

(B) We have $LK(n) \simeq K(n)$ for all $0 \leq n < \infty$.

(C) There exists an integer $n \geq 0$ such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$.

In case (A), Lemma 2 guarantees that L annihilates every finite p-local spectrum of type ≥ 0. In particular, for every X we have

$$LX \simeq X \otimes LS_{(p)} \simeq X \otimes 0 \simeq 0 :$$

that is, L is the zero functor.

Let us now analyze case (C). Fix n such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$. Lemma 4 implies that $\ker(L)$ contains every finite spectrum of type $> n$. Conversely, if X is a finite p-local spectrum such that $LX \simeq 0$, we have

$$0 \simeq K(n) \otimes LX \simeq LK(n) \otimes X \simeq K(n) \otimes X$$

so that X must have type $> n$. In other words, the finite p-local spectra belonging to $\ker(f)$ are *precisely* the spectra of type $> n$: that is, the spectra which are $E(n)$-acyclic. Conversely, we have the following:

Proposition 5. *Let* L *be a smashing localization, and suppose that* $LK(n) \simeq K(n)$. *Then every spectrum which belongs to* $\ker(L)$ *is* $E(n)$*-acyclic.*

Remark 6. An equivalent formulation is the following: if L is a smashing localization with $LK(n) \simeq K(n)$, then every $E(n)$-local spectrum is L-local.

Proof. Let $X \in \ker(L)$. We wish to show that X is $E(n)$-acyclic. Since $E(n)$ is Bousfield equivalent to $K(0) \oplus \cdots \oplus K(n)$, it suffices to show that X is $K(m)$-acyclic for $m \leq n$. This follows from

$$K(m) \otimes X \simeq LK(m) \otimes X \simeq K(m) \otimes LX \simeq 0,$$

since L is smashing and $LK(m) \simeq K(m)$ for $m \leq n$ (Lemma 3). □

Let us now return to case (C). If L is a smashing localization with $LK(n) \simeq K(n)$ and $LK(n+1) \simeq 0$, then we conclude that $\ker(L)$ consists of $E(n)$-acyclic spectra, and contains all finite E_n-acyclic spectra. In other words, we have

$$\ker(L_n^t) \subseteq \ker(L) \subseteq \ker(L_{E(n)}).$$

The following conjecture of Ravenel is the main open problem left in the subject (though it is generally believed to be false):

Conjecture 7 (Telescope Conjecture)**.** The localization functors L_n^t and $L_{E(n)}$ coincide. In particular, every smashing localization L satisfying (C) above has the form L_n^t for some $n \geq 0$.

It remains to treat the case (B): suppose that L is a smashing localization with $LK(n) \simeq K(n)$ for $n \geq 0$. According to Remark 6, if X is an $E(n)$-local spectrum for any X, then X is L-local. In particular, the *chromatic tower*

$$\cdots \to L_{E(2)}S_{(p)} \to L_{E(1)}S_{(p)} \to L_{E(0)}S_{(p)}$$

consists of L-local spectra, so that homotopy inverse limit of this tower is L-local. Next week we will prove the following:

Theorem 8 (Chromatic Convergence Theorem). *The homotopy inverse limit of the chromatic tower is $S_{(p)}$.*

Corollary 9. *Let L be a smashing localization such that $LK(n) \simeq K(n)$ for $0 \leq n < \infty$. Then L is equivalent to the identity functor.*

Proof. Using the chromatic convergence theorem and Remark 6, we deduce that $S_{(p)}$ is L-local. Then, for any p-local spectrum X, we have

$$LX \simeq X \otimes LS_{(p)} \simeq X \otimes S_{(p)} \simeq X.$$

□